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Unified Treatment for a Class of Extreme Value Distributions

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In this paper, we introduce a new family depends on Fréchet distribution. A sub-model of the new family called the composed- Fréchet exponential (C-FE) distribution is presented to provide the flexibality of the family. The point and interval estimation based on maximum likelihood are proposed. We also obtain the Bayes estimates of the unknown parameters under the assumption of independent gamma priors. The Bayes estimates of the unknown parameters cannot be obtained in a closed form. So, Markov Chain Monte Carlo (MCMC) method has been used to compute the approximate Bayes estimates under the squared error loss function and also the highest posterior density (HPD) intervals have been constructed. To examine the performance of our generated models in practice we use two real sets of data; then comparing the fitting of a new produced model with some well-known models, which provides the best fit to all of the data. .Further, a simulation study has been conducted to compare the performances of the Bayes estimators with corresponding maximum likelihood estimators.

keywords: Maximum likelihood estimation, Bayesian estimation, Monte Carlo Markov chain, Metropolis Hastings algorithm.

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1 Introduction

Statistical distributions are very useful in describing and predicting real world phenomena. Adding parameters to a well-established distribution is an effective way to enlarge the behavior range of this distribution and to obtain more flexible family of distributions to model various types of data. In the last few years, several ways of generating new probability distributions from classic ones were developed and discussed. Some well-known generalized classes (or generators) are odd exponentiated half-logistic-G family of distributions introduced by Afify et al. (2016) , Transformed-transformer (T-X) by Alzaatreh et al. (2013) , Kumaraswamy-G family of distributions introduced by Cordeiro and De Castro (2011) , the general rank transmutation (GRT) defined by Shaw and Buckley (2009) and the beta generalized family of distributions by Eugene et al. (2002).

The Fréchet (Fr) model is one of the most important distributions in modeling extreme values. The Fr model was originally proposed by Fréchet (1927). It has many applications in ranging, accelerated life testing, earthquakes, the floods, the wind speeds, the horse racing, the rainfall, queues in supermarkets and sea waves. One can find more details about the Fr model in the literature for example:Mead (2014) introduced Kumaraswamy Fr distribution. Mahmoud and Mandouh (2013) introduced transmuted Fr distribution. Barreto-Souza et al. (2011) introduced beta Fr distribution.Zaharim et al. (2009) applied the Fr distribution for analyzing the wind speed data.Nadarajah and Kotz (2008) discussed the sociological models based on Fr random variables (RVs). Nadarajah and Kotz (2003) investigated the exponentiated Fr distribution.

A random variable X is said to have the Fr distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given respectively by

$$g(x; a, b) = abx^{-a-1}e^{-bx^{-a}}, \quad x \geq 0,\ a, b > 0. \tag{1}$$

$$G(x; a, b) = e^{-bx^{-a}}. \tag{2}$$

Fattah and Ahmed (2018) proposed a new method for generating families of continuous distributions, called the composed -G Q family or shortly (C-G Q) family. This family based on the star-shaped property. They illustrated a set of important results in the theoretical reliability hold for a new family. These results make the family more important in applications.

Definition: Composed − G Q family

Let G and Q be two arbitrary continuous cumulative distribution functions of non-negative absolutely continuous random variables, G be strictly increasing on its support, and $G(0)$=$Q(0)$. Now define a cumulative distribution function (cdf), F, out of G and Q (called the composed- G Q family shortly (C- G Q)) as follows [see Fattah and Ahmed (2018)]:

$$F(x) = G(x, Q(x)), \quad \forall x \tag{3}$$

The corresponding probability density function (pdf) is given by

$$f(x) = g(x.Q(x))(xq(x) + Q(x)) \tag{4}$$

Where g and q are the corresponding densities of G and Q, respectively.
The rationale of using composed- G Q family is that this family has many features, such as this family is much richer in applications and the most important of these features is that it based on the star-shaped property which grantees the existences of some well know properties for the generated classes and distributions for any non-negative random variables. We will show a set of important results in the theoretical reliability hold for our newly introduced family. These results make the family much richer in applications. These results extracted from Barlow (1975) and are listed below for convenience.

Let F and G be continuous distributions, G be strictly increasing on its support, and $F(0) = G(0) = 0$. Then, F is star-shaped with respect to G (written $\underset{*}{<} G$) if $G^{-1}(F(x))$ is star-shaped,

$$\text{that is, } \left(\frac{1}{x}\right) G^{-1}(F(x)) \text{ is increasing for } x \geq 0.$$

Then:

a. $F \underset{c}{<} G$ implies $F \underset{*}{<} G$, (where $\underset{c}{<}$ implies the convex ordering).

b. The relationship $F \underset{c}{<} G$ is unaffected by a translation transformation of either F and G, assuming the random variables remain non-negative.

c. The relationship $F \underset{*}{<} G$ may be destroyed by a translation transformation of either F and G, assuming the random variables remain non-negative.

d. Let $G(x) = 1 - e^{-\lambda x}$, with $F(0) = 0$, then $F \underset{c}{<} G$, is equivalent to F having an increasing failure rate (IFR).

e. Let $G(x) = 1 - e^{-\lambda x}$, with $F(0) = 0$, then $F \underset{c}{<} G$, is equivalent to F having an increasing failure rate (IFR).

The Single Crossing Property. Let $F \underset{*}{\leq} G$, then

i. $\overline{F}(x)$ crosses $\overline{G}(\theta x)$ by one, at most, and from above, as x increases from 0 to ∞, for each $\theta > 0$.

ii. If, in addition, F and G have the same mean, then a single crossing does occur, and F has smaller variance then G.

iii. If we take G to be exponential distribution, then F must be IFRA by the previous results.

To this end, we present the following arguments.

We can see that the new generator enjoys the star-shaped property, which means any distribution derived based on the new family enjoys the results form a. to e.

Suppose that $G(x)$ and $Q(x)$ are the CDF'of the exponential distribution and are respectively given by $G(x;\lambda) = 1-e^{-\lambda x}$ and $Q(x;\beta) = 1-e^{-\beta x}$ (for $x > 0$ and $\beta, \lambda > 0$).

Then, a new distribution called composed-exponential exponential (C-EE), can be derived based on (3), and its CDF is given by

$$F(x;\beta,\lambda) = G(x \cdot Q(x)) = 1 - e^{-\lambda x \cdot \left(1-e^{-\beta x}\right)}, \quad x > 0,\ \beta, \lambda > 0$$

while, its corresponding pdf is given by

$$f(x;\beta,\lambda) = \lambda e^{-\lambda x \cdot \left(1-e^{-\beta x}\right)} \left(1 + (\beta x - 1)e^{-\beta x}\right)$$

Now, we check the existence of the star-shaped property for the new generated model C-EE.

i. For any given values of β, λ, and θ then $\overline{F}(x)$ crosses $\overline{G}(\theta x)$ at most one, and from above, as x increases from 0 to ∞, for $\theta > 0$.

While $\overline{F}(x)$ and $\overline{G}(\theta x)$ are respectively the survival functions of C–EE and exponential distribution, which are respectively given by $\overline{F}(x) = e^{-\lambda x \cdot \left(1-e^{-\beta x}\right)}$ and $\overline{G}(\theta x) = e^{-\lambda \theta x}$.

While $\overline{F}(x)$ and $\overline{G}(\theta x)$ are respectively the survival function of $\mathrm{C-EE}$ and exponential distribution which are respectively given by $\overline{F}(x) = e^{-\lambda x.(1-e^{-\beta x})}$ and $\overline{G}(\theta x) = e^{-\lambda \theta x}$. Figure 1 below shows this property visually for a given values for the unknown parameters.

Figure 1 show that $\overline{F}(x)$ crosses $\overline{G}(\theta x)$ at most one for a different values of β, λ and θ.

ii. Let $\lambda = 1.5$, $\beta = 3.2$ and $\theta = 0.826507$, then F and G have the same mean, so a single crossing does occur, and F has smaller variance than G. Figure 2 below shows this property visually for a given values for the unknown parameters.

At $\lambda = 1.5$, $\beta = 3.2$ and $\theta = 0.826507$, a single crossing does occur and the variance of F is 0.649751, while the variance of G at same values is 0.402951. It's clearly that F have smaller variance than G.

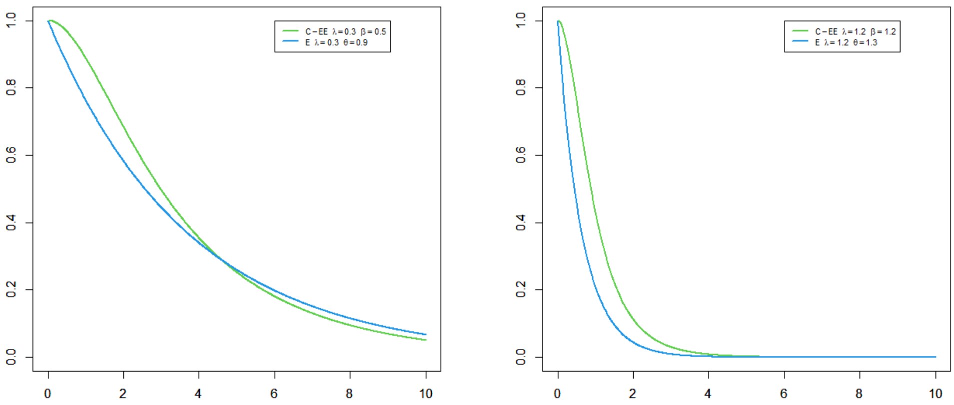
iii. If G be exponential distribution, then F must be IFRA

The cumulative hazard rate functions of the composed-exponential exponential (C-EE) is given by

$$H(x) = -\ln \overline{F}(x) = \lambda x \left(1 - e^{-\beta x}\right)$$

where $\overline{F}(x)$ is the survival function of C-EE distribution. Then,

$$\frac{H(x)}{x} = \frac{\lambda x \left(1 - e^{-\beta x}\right)}{x} = \lambda \left(1 - e^{-\beta x}\right)$$



(a) (b)

Figure 1: (a) and (b) The $\overline{F}(x)$ and $\overline{G}(\theta x)$ at different parameter values.

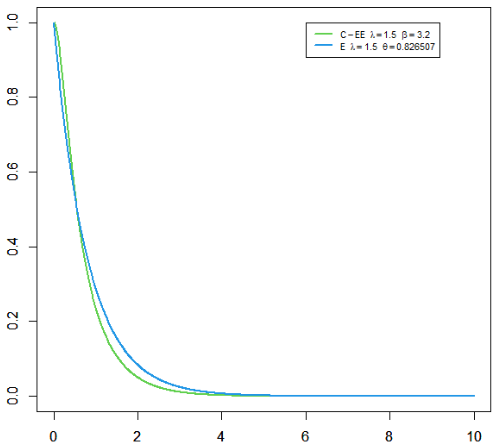


Figure 2: The $\overline{F}(x)$ and $\overline{G}(\theta x)$ at $\lambda = 1.5$, $\beta = 3.2$ and $\theta = 0.826507$ $\lambda = 1.5$, $\beta = 3.2$ and $\theta = 0.826507$

for λ, $\beta > 0$ the quantity $e^{-\beta x}$controls the behavior of the function $\frac{H(x)}{x}$, while $e^{-\beta x}$is a decreasing function, so $1 - e^{-\beta x}$ is an increasing function, then we can conclude that F is IFRA.
The main aim of this paper is to propose and study a new extention of the Fréchet distribution based on the method of composed-G Q family (C-G Q). The main purpose of the new model is that the additional parameter can give several desirable properties and more flexibility in the form of the hazard and density functions. This new distribution called composed Fréchet- exponential distribution (C-FE), which may be useful in representing the extreme values. The rest of the paper is organized as follows. In Section 2, The Composed-Fréchet Generated family and sub-model of the new generator called the composed Fréchet- exponential (C-F E) distribution are proposed. There is a demonstration of the graphs of the probability density function (pdf) and cumulative distribution function (cdf) of C-FE. In Section 3, the survival and hazard functions are studied. In Section 4, the statistical properties are obtained. In Section 5 the MLEs are obtained for the unknown parameters. In Section 6, the Bayes estimators of the unknown parameters using MCMC are introduced. In Section 7, we compare the performance of MLE and Bayesian estimates based on simulation studies. While applications to real-life data sets are provided in Section 8. Finally, in Section 9, the paper is concluded.

2 The Composed-Fréchet Generated Family

Suppose G is the cdf of Fréchet distribution, which given in (1), a new Fréchet family can be introduced using (3) and (4). This family will be named the composed Fréchet Q family (C-Fréchet Q (C-FQ)) and its cdf is given by

$$F(x; a,\ b,\ \lambda) = e^{-b(x \cdot Q(x))^{-a}} \tag{5}$$

with corresponding pdf

$$f(x; a,\ b,\ \lambda) = ab\,(x \cdot Q(x))^{-a-1}\, e^{-b(x \cdot Q(x))^{-a}} \cdot \{x \cdot q(x) + x \cdot Q(x)\} \tag{6}$$

2.1 The Composed-Fréchet Exponential Distribution

Substituting $q(x;\ \lambda) = \lambda e^{-\lambda x}$, and accordingly $Q(x,\ \lambda) =\ 1 - e^{-\lambda x}$ (where $x,\ \lambda > 0$) into (5) and (6), one gets the composed- Fréchet exponential (C-FE) distribution with cdf

$$F(x; a,\ b,\ \lambda) = e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}} \tag{7}$$

and its corresponding pdf is

$$f(x; a,\ b,\ \lambda) = ab\left(x(1 - e^{-\lambda x})\right)^{-a-1} e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}} \cdot \left\{x \cdot \lambda e^{-\lambda x} + 1 - e^{-\lambda x}\right\} \tag{8}$$

Figure (3) illustrates plots of the pdf and cdf of C-FE distribution for selected values of the parameters.

Figure 3 illustrates the plots of the pdf and cdf of C-FE distribution for different values of the parameters, which the plot of the pdf show that the distribution is unimodal. Thus $f(x)$ has a flat and relatively long right-hand tail.

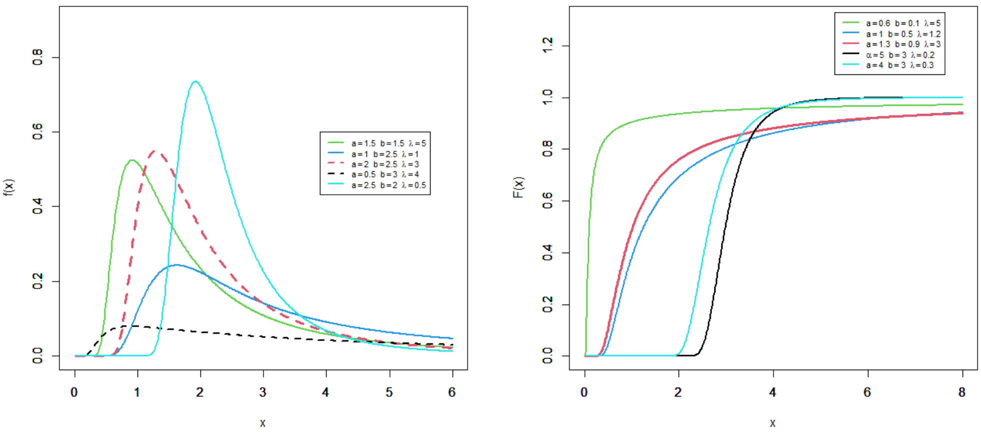


Figure 3: Plots of pdf and cdf of C-FE model for some parameters values.

3 The Survival and Hazard Functions

The survival and hazard functions are important for lifetime modeling in reliability studies. These functions are used to measure failure distributions and predict reliability lifetimes. The survival, hazard and reverse hazard functions of the C-FE distribution are defined respectively, by

$$S(x,\ a,\ b,\ \lambda) = 1 - F(x) = 1 - e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}} \tag{9}$$

$$h(x,\ a,\ b,\ \lambda) = \frac{f(x)}{1-F(x)} = \frac{ab\left(x(1-e^{-\lambda x})\right)^{-a-1}.\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\}}{e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}}-1}, \tag{10}$$

and

$$r(x) = \frac{f(x)}{F(x)} = ab\left(x(1-e^{-\lambda x})\right)^{-a-1}.\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\} \tag{11}$$

Figure (4) illustrates plots of the survival function of C-FE distribution for selected values of the parameters.

Figure (5) illustrates plots of the hazard function of C-FE distribution for selected values of the parameters.

The behavior of h(x) was studied for different values of parameters a and fixed b=1 and $\lambda = 2$ by Glaser's lemma (Glaser (1980)) following:

- If $a \leq 1$, then $h(x)$ has a decreasing failure rate which was satisfied in Figure (5c).
- If $a > 1$, then $h(x)$ has upside down bathtub failure rate which was satisfied in Figure (5d).

Figure (6) illustrates the plots of the reversed hazard function of C-FE distribution for selected values of the parameters.

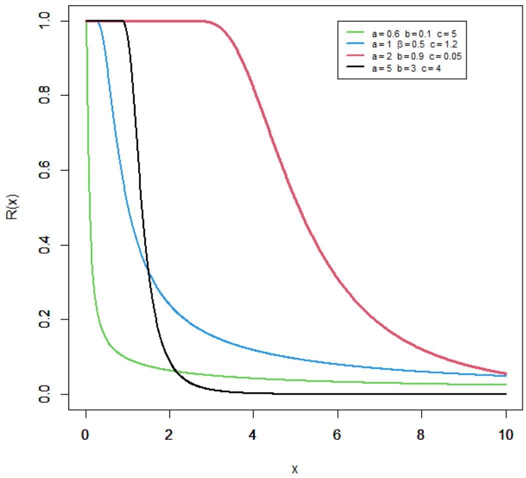
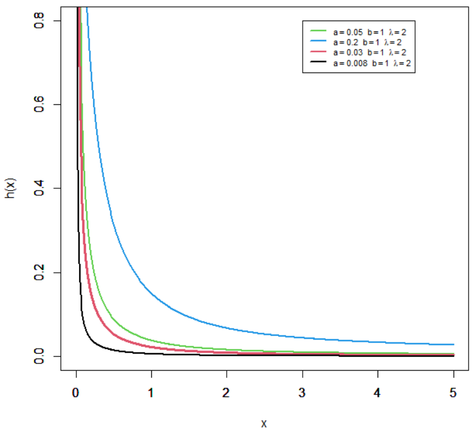
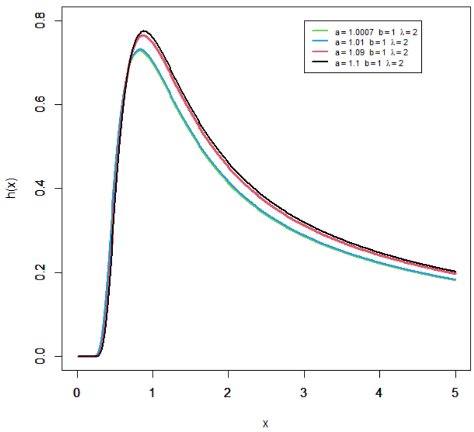


Figure 4: Plots of survival function of C-FE model for some parameters values.



(c)



(d)

Figure 5: Plots of hazard function of C-FE model for some parameters values.

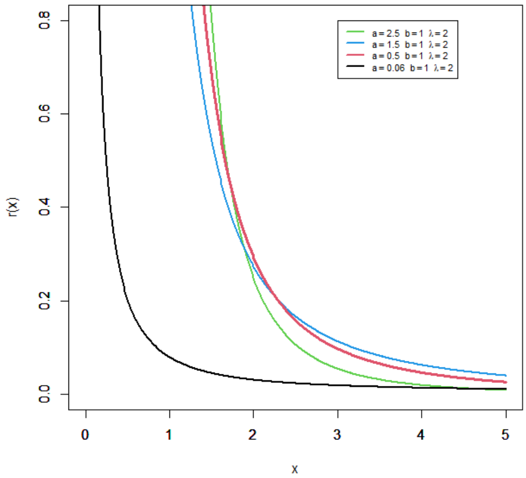


Figure 6: Plots of reversed hazard function of C-FE model for some parameters values.

4 Statistical Properties

This section explains the statistical properties of the C-FE distribution in general terms.

4.1 Moments

For a C-FE random variable X, the r^{th}order moment about zero are given by

$$E\left(X^r\right) = \sum_{i=k=0}^{\infty} C_{ik} \frac{\Gamma(r - a(i+1))}{\lambda^{r-a(i+1)}} \left[\frac{1}{k^{r-a(i+1)}} + \frac{r - a(i+1)}{(1+k)^{r-a(i+1)}}\right] \tag{12}$$

Where,

$$C_{ik} = (-1)^{i+k} a b^{i+1} \binom{-a(i+1)-1}{k}$$

Proof
See Proof 1 in appendix.

4.2 Moment Generating Function

The moment generating function (mgf) of the C-FE random variable X is obtained as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(x^r\right)$$

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=k=0}^{\infty}(-1)^{i+k}ab^{i+1}\frac{t^r}{r!}\binom{-a(i+1)-1}{k}\frac{\Gamma(r-a(i+1))}{\lambda^{r-a(i+1)}}\cdot$$
$$\cdot\left[\frac{1}{k^{r-a(i+1)}}+\frac{r-a(i+1)}{(1+k)^{r-a(i+1)}}\right].$$

4.3 Quantile Function

$$F^{-1}(.) = U$$

$$x_q\left(1-e^{-\lambda x_q}\right)+\left(\frac{b}{\log(u)}\right)^{\frac{1}{a}} = 0 \tag{13}$$

where, $X \sim U(0,\ 1)$

The above equation has no closed form solution in x_q, so one has to use a numerical technique such as a Newton- Raphson method to get an approximate value of the quantile. If we put q=0.5 we will get the median of C-FE distribution. As we will be seen in Section 6.

4.4 Order Statistics

Let $X_{1;n},\ X_{2;n},\ X_{3;n},\ldots,\ X_{i;n}$ be the order statistics of a random sample $X_1,\ X_2, X_3,\ \ldots,\ X_n$ observed from C-FE distribution with cdf $F(x)$ and pdf $f(x)$then the density function of $X_{i;n}$ is given by

$$f_{i;n}(x) = \frac{n!}{(i-1)!(n-i)!}k_{rj}ab^{r+1}\left(x\left(1-e^{-\lambda x}\right)\right)^{-a(1+r)-1}\cdot\left\{x\cdot\lambda e^{-\lambda x}+1-e^{-\lambda x}\right\} \tag{14}$$

Where,

$k_{rj} = \sum_{r=j=0}^{\infty}\frac{(-1)^{j+r}}{r!}\binom{n-i}{j}(i+j)^r.$

proof

See Proof 2 in appendix.

4.5 Stochastic Ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A continuous random variable X is said to be smaller than a continuous random variable Y in the

(i) Stochastic order ($x \leq_{st} y$) if $F_X(x) \geq F_Y(x) for\ all\ x.$

(ii) Hazard rate order ($x \leq_{hr} y$) if $h_X(x) \geq h_Y(x) for\ all\ x.$

(iii) Likelihood ratio order ($x \leq_{Lr}\ y$) if $\frac{F_X(x)}{F_Y(x)}$ decreases in x.

C-FE distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following

Suppose random variables X and Y are distributed according to C-FE $(a_1,\ b_1,\ \lambda_1)$ and C-FE $(a_2,\ b_2,\ \lambda_2)$. Then, the following results hold true. If $b_1 = b_2,\ \ \lambda_1 = \lambda_2 \quad and\ a_2 < a_1, \quad then\ x \leq_{Lr} y, \quad x \leq_{hr} y\ and\ x \leq_{st} y.$

$$\frac{\partial}{\partial x}\log\frac{f_X(x)}{f_Y(y)} = -\frac{(a_1+1)k(x;\lambda_1)}{\left(x\left(1-e^{-\lambda_1 x}\right)\right)} + \frac{(a_2+1)e(x;\lambda_2)}{\left(x\left(1-e^{-\lambda_2 x}\right)\right)} + a_1 b_1\left(k\left(x;\lambda_1\right)\right)\left(x\left(1-e^{-\lambda_1 x}\right)\right)^{-a_1-1}$$
$$-a_2 b_2\left(e\left(x;\lambda_2\right)\right)\left(x\left(1-e^{-\lambda_2 x}\right)\right)^{-a_2-1}$$
$$+\left[\frac{(g(x;\lambda_2))\left(2\lambda_1 e^{-\lambda_1 x}-\lambda_1{}^2 x e^{-\lambda_1 x}\right)-(h(x;\lambda_1))\left(2\lambda_2 e^{-\lambda_2 x}-\lambda_2{}^2 x e^{-\lambda_2 x}\right)}{(h(x;\lambda_1))(g(x;\lambda_2))}\right].$$

Proof: We have

$$\frac{f_X(x)}{f_Y(y)} = \frac{a_1 b_1\left(x(1-e^{-\lambda_1 x})\right)^{-a_1-1} e^{-b_1\left(x(1-e^{-\lambda_1 x})\right)^{-a_1}}.\{x.\lambda_1 e^{-\lambda_1 x}+1-e^{-\lambda_1 x}\}}{a_2 b_2\left(x(1-e^{-\lambda_2 x})\right)^{-a_2-1} e^{-b_2\left(x(1-e^{-\lambda_2 x})\right)^{-a_2}}.\{x.\lambda_2 e^{-\lambda_2 x}+1-e^{-\lambda_2 x}\}}$$

Thus,

$$\log\frac{f_X(x)}{f_Y(y)} = \log\left(\frac{a_1 b_1}{a_2 b_2}\right) - (a_1+1)\log\left(x\left(1-e^{-\lambda_1 x}\right)\right) + (a_2+1)\log\left(x\left(1-e^{-\lambda_2 x}\right)\right)$$

$$-b_1\left(x\left(1-e^{-\lambda_1 x}\right)\right)^{-a_1} + b_2\left(x\left(1-e^{-\lambda_2 x}\right)\right)^{-a_2} + log\left(\frac{x.\lambda_1 e^{-\lambda_1 x}+1-e^{-\lambda_1 x}}{x.\lambda_2 e^{-\lambda_2 x}+1-e^{-\lambda_2 x}}\right)$$

$$\frac{\partial}{\partial x}\log\frac{f_X(x)}{f_Y(y)} = -\frac{(a_1+1)(k(x;\lambda_1))}{\left(x\left(1-e^{-\lambda_1 x}\right)\right)} + \frac{(a_2+1)k(x;\lambda_2)}{\left(x\left(1-e^{-\lambda_2 x}\right)\right)} + a_1 b_1\left(k\left(x;\lambda_1\right)\right)\left(x\left(1-e^{-\lambda_1 x}\right)\right)^{-a_1-1}$$
$$-a_2 b_2\left(k\left(x;\lambda_2\right)\right).\left(x\left(1-e^{-\lambda_2 x}\right)\right)^{-a_2-1}$$
$$+\left[\frac{h(x;\lambda_2)}{h(x;\lambda_1)}\right]\left[\frac{h(x;\lambda_2)\left(2\lambda_1 e^{-\lambda_1 x}-\lambda_1{}^2 x e^{-\lambda_1 x}\right)-h(x;\lambda_1)\left(2\lambda_2 e^{-\lambda_2 x}-\lambda_2{}^2 x e^{-\lambda_2 x}\right)}{(h(x;\lambda_2))^2}\right].$$

Now if $b_1 = b_2, \quad \lambda_1 = \lambda_2 \quad and\ a_2 < a_1, \quad then\ \frac{\partial}{\partial x}\log\frac{f_x(x)}{f_y(y)} \leq 0$, which implies that $x \leq_{Lr} y, \quad x \leq_{hr} y\ and\ x \leq_{st} y.$

Where,

$h\left(x;\lambda_i\right) = x.\lambda_i e^{-\lambda_i x} + 1 - e^{-\lambda_i x}$, $k\left(x;\lambda_i\right) = 1 + \lambda_i x e^{-\lambda_i x} - e^{-\lambda_i x}$.

and $i = 1, 2$.

5 Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the unknown parameters of the C-FE distribution from complete samples only.

Let $x_1,\ x_2,\ \ldots,\ x_n$ be a random sample of size n from C-FE $(x;\varnothing), \quad \varnothing = (a,\ b,\ \lambda)$. The log likelihood function for the vector of parameters $\varnothing = (a,\ b,\ \lambda)$ can be written as

$$logl(x) = \; n\ln(a) + n\ln(b) - (a+1)\sum_{i=1}^{n}\ln\left(x_i\left(1-e^{-\lambda x_i}\right)\right) - b\sum_{i=1}^{n}\left(x_i\left(1-e^{-\lambda x_i}\right)\right)^{-a}$$

$+\sum_{i=1}^{n}\ln\left\{x_i.\lambda e^{-\lambda x_i}+1-e^{-\lambda x_i}\right\}.$

By taking the partial derivatives of the log-likelihood function with respect to a, b and λ, we obtain the components of the score vector as follows

$$\frac{\partial \ln L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\ln\left(u\left(x_i;\lambda\right)\right) + b\sum_{i=1}^{n} v\left(x_i;a,\lambda\right)\ln\left(u\left(x_i;\lambda\right)\right) \tag{15}$$

$$\frac{\partial \ln L}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n} v\left(x_i;a,\lambda\right) \tag{16}$$

$$\frac{\partial \ln L}{\partial \lambda} = -(a+1)\sum_{i=1}^{n}\frac{x_i^2 e^{-\lambda x_i}}{u\left(x_i;\lambda\right)} + ab\sum_{i=1}^{n} x_i^2 e^{-\lambda x_i}\frac{v\left(x_i;a,\lambda\right)}{u\left(x_i;\lambda\right)} + \frac{2e^{-\lambda x_i} - \lambda x_i^2 e^{-\lambda x_i}}{c\left(x_i;\lambda\right)} \tag{17}$$

Where,

$$v\left(x_i;a,\lambda\right) = u^{-a}\left(x_i;\lambda\right) = \left[x_i\left(1-e^{-\lambda x_i}\right)\right]^{-a},$$

$$u\left(x_i;\lambda\right) = x_i d\left(x_i;\lambda\right),$$

$$d\left(x_i;\lambda\right) = \left(1-e^{-\lambda x_i}\right),$$

$$c\left(x_i;\lambda\right) = x_i.\lambda e^{-\lambda x_i} + d\left(x_i;\lambda\right).$$

and $i = 1, \ldots, \; n$

The maximum likelihood estimate of b, say $\widehat{b}$ can be obtained by solving Equation (16) as

$$\widehat{b} = \frac{n}{\sum_{i=1}^{n} v\left(x_i;\widehat{a},\widehat{\lambda}\right)}$$

The MLEs of a and λ can be determined numerically from the solution of nonlinear system of Equations (15) and (17); subsequently, these solutions will yield the MLE estimators $\widehat{a}, \widehat{\lambda}$.

To construct asymptotic confidence intervals, we need to obtain the observed Fisher information matrix. which is given by

$$\widehat{F} = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial a^2} & \frac{\partial^2 \ln L}{\partial a \partial b} & \frac{\partial^2 \ln L}{\partial a \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial a \partial b} & \frac{\partial^2 \ln L}{\partial b^2} & \frac{\partial^2 \ln L}{\partial b \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial a \partial \lambda} & \frac{\partial^2 \ln L}{\partial b \partial \lambda} & \frac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}_{\varnothing=\widehat{\varnothing}} \tag{18}$$

Where,
$\frac{\partial^2 lnL}{\partial a^2} = \frac{n}{a^2} - b\sum_{i=1}^{n} v(x_i; a, \lambda)\left(\ln\left(u(x_i; \lambda)\right)\right)^2,$
$\frac{\partial^2 lnL}{\partial b^2} = \frac{-n}{b^2},$
$\frac{\partial^2 lnL}{\partial \lambda^2} = (a+1)\sum_{i=1}^{n}\left[\frac{{x_i}^3 e^{-\lambda x_i}\ u(x_i;\lambda)+{x_i}^4 e^{-2\lambda x_i}}{u^2(x_i;\lambda)}\right] - ab\sum_{i=1}^{n}\left[a{x_i}^3 e^{-2\lambda x_i}\frac{v(x_i;a,\lambda)}{u(x_i;\lambda)}\right]$
$+\ \frac{c(x_i;\lambda)\left(\lambda {x_i}^3 e^{-\lambda x_i} - 2x_i e^{-\lambda x_i} - {x_i}^2 e^{-\lambda x_i}\right) - \left(2e^{-\lambda x_i} - \lambda {x_i}^2 e^{-\lambda x_i}\right)\left(\lambda {x_i}^2 e^{-\lambda x_i} + 2x_i e^{-\lambda x_i}\right)}{\left(c(x_i;\lambda)\right)^2},$
$\frac{\partial^2 lnL}{\partial a \partial b} = \frac{n}{a} - \frac{n}{b} - \sum_{i=1}^{n} v\left(x_i; a, \lambda\right)\ln\left(u\left(x_i; \lambda\right)\right),$
$\frac{\partial^2 lnL}{\partial b \partial \lambda} = \frac{n}{a} - \sum_{i=1}^{n}\frac{{x_i}^2 e^{-\lambda x_i}}{u(x_i;\lambda)} + ab{x_i}^2 e^{-\lambda x_i}\frac{v(x_i;a,\lambda)}{u(x_i;\lambda)} + \frac{2e^{-\lambda x_i} - \lambda {x_i}^2 e^{-\lambda x_i}}{c(x_i;\lambda)}.$

Obtaining the inverse of the matrix $\widehat{F}$, which we dented by $\widehat{V}$, provides the asymptotic variance co-variances matrix for $\varnothing = (a,\ b,\ \lambda)$. Assume that the regularity condition is satisfied, use (18) to get a 100 (1-γ) % confidence intervals for the parameters a, b and λ as follows
$\widehat{a} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V}_{11}}$, $\widehat{b} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V}_{22}}$, $\widehat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V}_{33}}$.
Where $z_{\frac{\gamma}{2}}$ is the upper $\gamma^{th}$of the standard normal distribution.

6 Bayesian Estimation

In this section, the MCMC algorithm for computing the Bayes estimates of parameters a, b and λ of the C-FE distribution is used. MCMC is one of the best techniques for obtaining the Bayes estimates, for more details about the MCMC methods, see, e.g., Gelfand and Smith (1990) , Casella and Robert (2008) and Upadhyay and Gupta (2010). The Metropolis-Hastings algorithm (MH) is used, to generate samples from the conditional posterior distributions, and then we compute the Bayes estimates. Assume that a, b and λ are independent and have prior distributions π_1, π_2, π_3 respectively. Prior for each a, b and λ is assumed to follow a gamma distribution. We use gamma distribution as prior distribution because Abbas et al. (2013) used the gamma distribution as a prior distribution in Bayesian estimation for Fréchet distribution and this will be more appropriate in Bayesian estimation for C-FE distribution.

$$\pi_1(a) = \frac{{m_1}^{q_1} a^{q_1-1} e^{-m_1 a}}{\Gamma(q_1)} \qquad a > 0,\ m_1,\ q_1 > 0$$

$$\pi_2(b) = \frac{{m_2}^{q_2} b^{q_2-1} e^{-m_2 b}}{\Gamma(q_2)} \qquad b > 0,\ m_2,\ q_2 > 0$$

and

$$\pi_3(\lambda) = \frac{{m_3}^{q_3} \lambda^{q_3-1} e^{-m_3 \lambda}}{\Gamma(q_3)} \qquad \lambda > 0,\ m_3,\ q_3 > 0$$

Here, hyper-parameters $m_1,\ q_1,\ m_2,\ q_2,\ m_3$and q_3 are chosen to reflect the prior knowledge about the unknown parameters. Suppose that we have n number of samples available from C-FE distribution, and the maximum likelihood estimates of $(a,\ b,\ \lambda)$ are $\left(\widehat{a}^j,\ \widehat{b}^j,\ \widehat{\lambda}^j\right)$, $j = 1,\ 2,\ 3,\ \ldots,\ n$. Then equating the mean and variance of

$\left(\widehat{a}^j,\ \widehat{b}^j,\ \widehat{\lambda}^j\right)$ with the mean and variance of the suggested priors (gamma priors), we can get [see Dey et al (2016)]

$$\frac{1}{n}\sum_{j=0}^{n}\widehat{a}^j=\frac{q_1}{m_1},\quad \frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{a}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{a}^j\right)^2=\frac{q_1}{{m_1}^2} \tag{19}$$

$$\frac{1}{n}\sum_{j=0}^{n}\widehat{b}^j=\frac{q_2}{m_2},\quad \frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{b}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{b}^j\right)^2=\frac{q_2}{m_2^2} \tag{20}$$

and

$$\frac{1}{n}\sum_{j=0}^{n}\widehat{\lambda}^j=\frac{q_3}{m_3},\quad \frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{\lambda}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{\lambda}^j\right)^2=\frac{q_3}{m_3^2} \tag{21}$$

Solving Equations (19), (20) and (21), we get the estimated hyper-parameters as follows:

$b_1=\frac{\left(\frac{1}{n}\sum_{j=0}^{n}\widehat{a}^j\right)^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{a}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{a}^j\right)^2}$, $m_1=\frac{\left(\frac{1}{n}\sum_{j=0}^{n}\widehat{a}^j\right)^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{a}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{a}^j\right)^2}$,

$b_2=\frac{\left(\frac{1}{n}\sum_{j=0}^{n}\widehat{b}^j\right)^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{b}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{b}^j\right)^2}$, $m_2=\frac{\left(\frac{1}{n}\sum_{j=0}^{n}\widehat{b}^j\right)^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{b}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{b}^j\right)^2}$,

and

$b_3=\frac{\left(\frac{1}{n}\sum_{j=0}^{n}\widehat{\lambda}^j\right)^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{\lambda}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{\lambda}^j\right)^2}$, $m_3=\frac{\left(\frac{1}{n}\sum_{j=0}^{n}\widehat{\lambda}^j\right)^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\widehat{\lambda}^j-\frac{1}{n}\sum_{j=0}^{n}\widehat{\lambda}^j\right)^2}$.

Therefore, the joint prior distribution of a, b and λ can be written as

$$\pi(a,b,\lambda)=\frac{m_1^{q_1}a^{q_1-1}e^{-m_1a}}{\Gamma(q_1)}\cdot\frac{m_2^{q_2}b^{q_2-1}e^{-m_2b}}{\Gamma(q_2)}\cdot\frac{m_3^{q_3}\lambda^{q_3-1}e^{-m_3\lambda}}{\Gamma(q_3)}\quad a,b,\lambda>0,\quad m_i,b_i>0. \tag{22}$$

where i = 1,2,3

The joint posterior density function of a, b and λ can be written as

$$\pi\left(a,\ b,\lambda|x\right)=k\frac{{m_1}^{q_1}a^{n+q_1-1}e^{-m_1a}}{\Gamma\left(q_1\right)}\frac{{m_2}^{q_2}b^{n+q_2-1}e^{-m_2b}}{\Gamma\left(q_2\right)}\frac{{m_3}^{q_3}\lambda^{q_3-1}e^{-m_3\lambda}}{\Gamma\left(q_3\right)}$$

$\cdot\ \prod_{i=1}^{n}\left(x_i(1-e^{-\lambda x_i})\right)^{-a-1}.e^{-b\sum_{i=1}^{n}\left(x_i(1-e^{-\lambda x_i})\right)^{-a}}$

$$\prod_{i=1}^{n}\left\{x_i\cdot\lambda e^{-\lambda x_i}+1-e^{-\lambda x_i}\right\} \tag{23}$$

where k is the normalizing constant which is given by

$k^{-1}=\int_0^\infty\int_0^\infty\int_0^\infty\frac{{m_1}^{q_1}a^{n+q_1-1}e^{-m_1a}}{\Gamma(q_1)}\frac{{m_2}^{q_2}b^{n+q_2-1}e^{-m_2b}}{\Gamma(q_2)}\frac{{m_3}^{q_3}\lambda^{q_3-1}e^{-m_3\lambda}}{\Gamma(q_3)}$.

$\cdot\prod_{i=1}^{n}\left(x_i(1-e^{-\lambda x_i})\right)^{-a-1}.e^{-b\sum_{i=1}^{n}\left(x_i(1-e^{-\lambda x_i})\right)^{-a}}$

$\cdot \prod_{i=1}^{n} \{x_i.\lambda e^{-\lambda x_i} + 1 - e^{-\lambda x_i}\}\ dadbd\lambda.$

Therefore, the Bayes estimate of any function of a,b and λ, say $g(a,\ b,\ \lambda)$, under the squared error loss function, is given by

$\widetilde{g}(a,\ b,\ \lambda) = E_{a,\ b,\ \lambda|x}\left(g(a\ ,\ b\ \lambda)\right) = \int_0^\infty \int_0^\infty \int_0^\infty g(a\ ,\ b\ \lambda)\pi\left(a,\ b, \lambda|x\right)$

It is clear from Equation (23) that there is no closed form for the estimators, and, hence, MCMC procedure is suggested to compute the Bayes estimates. We consider the Metropolis-Hastings (M H) algorithm with a normal proposal distribution to generate samples from the conditional posterior distributions. The following is the used code to generate the required by samples M H algorithm:

1) Set initial value of ϕ as $\phi = \phi^{(0)}$, and set $i = 1$, where $\phi = (\ a,\ b,\ \lambda)$.
2) Set $\phi = \phi^{(i-1)}$.
3) Generate a proposal, $\varnothing^*$, following a multivariate normal, N($\phi,\ s_\phi$), where s_ϕ is the standard deviation (we suggest $s_\phi =\ (0.001, 0.003, 0.002)$.
4) *Calculate the acceptance probability,* $\tau = min\left(1, \frac{\pi(\phi^*|x)}{\pi(\phi|x)}\right)$.
5) *Generate* $U \sim U(0,\ 1)$.
6) $If \leq \tau,\ set\ \phi^{(i)} = \phi^*,\ otherwise,\ set\ \phi^{(i)} = \phi^{(i-1)}$.
7) Set $i = i + 1$.
8) *Repeat steps* 2 *to* 7 *by* N *times and obtain* $\phi^{(j)},\ j = 1, 2, \ldots, N$.

   After getting MCMC samples from the posterior distribution, we can find the Bayes estimate for the parameters in the following way:

$$\widehat{a} = \frac{1}{N - B} \sum_{i=B+1}^{N} a^{(i)},$$

$\widehat{b} = \frac{1}{N-B} \sum_{i=B+1}^{N} b^{(i)},$

and

$\widehat{\lambda} = \frac{1}{N-B} \sum_{i=B+1}^{N} \lambda^{(i)}.$

where B is the number of burn-in samples. Then we calculate the highest posterior density (HPD) intervals for the unknown parameters of the C-FE distribution using the method of Chen and Shao (1999).

One can also refer to Kundu and Pradhan (2009) and Dey and Dey (2014) for a review on this method. We will use the samples drawn using the proposed MH algorithm to construct the interval estimates. Let us assume that $\Pi\left(\phi|X\right)$

denotes the posterior distribution function of ϕ. Let us further suppose that $\phi^{(p)}$is the p^{th} quantile of ϕ, that is, $\phi^{(p)}= inf\{\lambda;\, \mathbf{\Pi}\,(\phi^{*}|\mathbf{X})\}$, where $0 <$ p < 1. Notice that for a given $\varnothing^{*}$a simulation consistent estimator of $\mathbf{\Pi}\,(\phi^{*}|\mathbf{X})$ can be estimated as

$$\Pi\,(\phi^{*}|X) = \frac{1}{N-B}\sum_{i=B+1}^{N} I_{\phi\leq\phi^{*}}$$

Here , $I_{\varnothing\leq\varnothing^{*}}$ is the indicator function. Then the corresponding estimate is obtained as

$$\widehat{\Pi}\,(\phi^{*}|X) = \begin{cases} 0 & if\ \phi^{*} <\ \phi_{B} \\ \sum_{j=B}^{i} w_j & if\ \phi_{(i)} < \phi^{*} <\ \phi_{(i+1)} \\ 1 & if\ \phi^{*} >\ \phi_{N} \end{cases}$$

where $w_j = \frac{1}{N-B}$ and $\phi_{(j)}$ are the ordered values of $\phi_{(j)}$.
Now, $i = B, \ldots, N$, $\phi^{(p)}$can be approximated by

$$\phi^{(p)} = \begin{cases} \varnothing_{(B)} & if\ p = 0 \\ \varnothing_{(i)} & if\ \sum_{j=B}^{i-1} w_j < p < \sum_{j=B}^{i} w_j. \end{cases}$$

Now, to obtain a 100 (1-p) % HPD credible interval for ϕ, let $R_j = \left(\widehat{\varnothing}^{\left(\frac{1}{N}\right)},\ \widehat{\varnothing}^{\left(\frac{j+(1-p)N}{N}\right)}\right)$ for j = $B \ldots$,[pN], here [a] denotes the largest integer less than or equal to a. Then choose R_j among all the $\grave{R_j}s$ such that it is smallest width.

7 Simulation Study

In this section, we compare the performances of MLEs and Bayesian estimates using the MCMC method. Sample of size {n=100, 150, 200, 250, 300, 350} are used to generate observations from a C-FE distribution with different true values. We assume that the number of repetition is 1000, then we calculate their means, means square errors (MSE) and associated 95% confidence interval (CI) of each parameter. For Bayesian estimations, the MCMC method will be used according to MH Algorithm. The number of iteration for this algorithm is N = 10000 with burn-in period $B = 2000$. Each prior for a, b and λ is assumed to be gamma distribution with hyper parameters ($b_1 = 12,\ m_1 = 5.8,\ b_2 = 14,\ m_2 = 5.6,\ b_3 = 1.5,\ m_3 = 0.41$).Then the Bayes estimates and HPD interval estimates are obtained using the technique of Chen and Shao (1999). The performances of the estimators for both methods using MSE and average interval lengths (AIL) with coverage percentages (CP) are reported in Tables (1-6). From Tables (1-6), we notice that;

i. The MSE of the Bayes estimates are better than their corresponding MSE of MLEs for all samples.

ii. The average bias of the Bayesian is less than that of the MLEs in all cases.

iii. The 95% Bayes intervals are smaller than the asymptotic confidence intervals of MLEs for all cases.

Table 1: Estimates Values and MSEs (With a = 10, b = 10, λ = 10)

n	Parameters	MLE			Bayesian		
		Parameters estimates	MSE	Bias	Bayes	MSE	Bias
100	a	10.0592	0.8224	0.0592	9.9945	0.0020	-0.0055
	b	10.2057	6.0938	0.2057	9.9648	0.0194	-0.0352
	λ	9.7361	2.7981	-0.2638	9.9973	0.0079	-0.0027
150	a	10.0249	0.6595	0.0249	9.9946	0.0020	-0.0054
	b	10.0665	5.4824	0.0664	9.9612	0.0193	-0.0388
	λ	9.6397	3.4994	-0.3603	10.0021	0.0073	0.0021
200	a	9.9221	0.5900	-0.0779	9.9999	0.0010	-0.00004
	b	9.7612	5.3199	-0.2388	10.0018	0.0093	0.0018
	λ	9.4499	4.7484	-0.5500	9.9986	0.0044	-0.0014
250	a	9.8771	0.6169	-0.1228	9.9994	0.0010	-0.0006
	b	9.5302	6.1636	-0.4698	9.9967	0.0087	-0.0032
	λ	9.2598	6.4087	-0.7402	9.9992	0.0043	-0.0008
300	a	9.9079	0.4598	-0.0920	9.9997	0.0009	-0.0003
	b	9.6605	4.4881	-0.3394	9.9963	0.0090	-0.0036
	λ	9.5180	4.6003	-0.4819	9.9990	0.0039	-0.0009
350	a	9.9108	0.4269	-0.0892	10.00002	0.0009	0.0003
	b	9.6659	4.4299	-0.3340	10.0035	0.0085	0.0035
	λ	9.4743	4.7914	-0.5256	9.9982	0.0039	-0.0018

8 Applications for Real Data

This section illustrates the applicability and flexibility of the C-FE distribution with two real data sets, which will represents as follow:

8.1 Data set (1): Minimum Monthly Flows of Water on The Piracicaba River

Two real data sets are represented related to minimum monthly flows of water (m^3/s) during May and August on the Piracicaba River, located in São Paulo state, Brazil. The data have been provided by Ramos et al. (2020) . The data is given by:

- May: 29.19, 18.47, 12.86, 151.11, 19.46, 19.46, 84.30, 19.30, 18.47, 34.12, 374.54, 19.72, 25.58, 45.74, 68.53, 36.04, 15.92, 21.89, 40.00, 44.10, 33.35, 35.49, 56.25,

Table 2: CI, HPD interval, AILs and CPs (With a = 10, b = 10, λ = 10).

n	Parameters	MLE			Bayesian		
		CI	AIL	CP	HPD Interval	AIL	CP
100	a	(7.8659,11.7429)	3.8771	97.5	(9.9050,10.0773)	0.1722	96.7
	b	(3.0632,15.2997)	12.2365	97.5	(9.6995,10.2321)	0.5325	97.6
	λ	(1.8844,11.1123)	9.2279	97.5	(9.8282,10.1713)	0.3430	97.4
150	a	(7.9771,11.4939)	3.5169	97.5	(9.9084,10.0805)	0.1721	97.9
	b	(2.7057,14.3136)	11.6079	97.5	(9.7044,10.2291)	0.5246	97.1
	λ	(1.8617,10.7465)	8.8848	97.5	(9.8481,10.1804)	0.3323	98.1
200	a	(7.9635,11.2306)	3.2671	97.5	(9.9435, 10.0693)	0.1258	98.9
	b	(2.2788,13.2935)	11.0147	97.5	(9.8049,10.1699)	0.3650	96.4
	λ	(1.7734,10.6124)	8.8390	97.5	(9.8802,10.1351)	0.2549	98.3
250	a	(7.7338,11.1738)	3.4399	97.5	(9.9554,10.0754)	0.1200	98.7
	b	(1.9327,12.9394)	11.0067	97.5	(9.8772,10.1776)	0.3005	97.5
	λ	(1.7604,10.5709)	8.8105	97.5	(9.8900,10.1336)	0.2436	98.2
300	a	(7.6897,10.8714)	3.1816	97.5	(9.9609,10.0669)	0.1060	98.9
	b	(1.7522,12.5755)	10.8233	97.5	(9.9487,10.1693)	0.2206	97.8
	λ	(1.6993,10.4682)	8.7689	97.5	(9.9007,10.1254)	0.2257	98.1
350	a	(7.5834,10.8223)	3.2389	97.5	(9.9515,10.0636)	0.1121	99.0
	b	(1.6319,12.2898)	10.6579	97.5	(9.9234,10.1743)	0.2509	97.9
	λ	(1.6234,10.3559)	8.7325	97.5	(9.9267,10.1109)	0.1846	98.0

Table 3: Estimate Values and MSEs (With a = 8.5, b = 10, λ = 9.5).

n	Parameters	MLE			Bayesian		
		Parameters estimates	MSE	Bias	Bayes	MSE	Bias
100	a	9.8789	2.9819	1.3789	8.5053	0.0019	0.0053
	b	9.0033	10.2896	-0.9967	9.9225	0.0237	-0.0775
	λ	8.5464	9.3405	-0.9536	9.5004	0.0079	0.0004
150	a	9.8299	2.6209	1.3299	8.5093	0.0019	0.0093
	b	8.8898	8.4721	-1.1102	9.9073	0.0252	-0.0927
	λ	8.6750	8.7508	-0.8249	9.4998	0.0076	-0.0002
200	a	9.8996	2.5517	1.3996	8.5002	0.0010	0.0002
	b	9.0778	6.1271	-0.9222	10.0019	0.0094	0.0019
	λ	8.8283	6.1922	-0.6717	9.4983	0.0039	-0.0017
250	a	9.8203	2.5193	1.3203	8.5016	0.0009	0.0016
	b	8.8340	8.2288	-1.1659	10.0014	0.0095	0.0014
	λ	8.5803	8.3164	-0.9197	9.4977	0.0039	-0.0023
300	a	9.7335	2.2051	1.2335	8.5004	0.0009	0.0001
	b	8.5284	9.1257	-1.4716	10	9.0096	0.0009
	λ	8.1313	11.3031	-1.3687	9.5029	0.0036	0.0029
350	a	9.7193	2.1737	1.2193	8.5025	0.0009	0.0025
	b	8.4686	9.3154	-1.5313	9.9973	0.0089	-0.0027
	λ	8.0989	11.7703	-1.4010	9.5007	0.0043	0.0007

Table 4: CI, HPD interval, AILs and CPs (With a = 8.5, b = 10, λ = 9.5).

n	Parameters	MLE			Bayesian		
		CI	AIL	CP	HPD Interval	AIL	CP
100	a	(7.4431, 11.6892)	4.2461	97.5	(8.4191, 8.5890)	0.1699	97.7
	b	(1.9392, 14.3083)	12.3691	97.5	(9.6604, 10.1845)	0.5240	97.6
	λ	(1.6515, 11.9980)	10.3465	97.5	(9.3209, 9.6557)	0.3347	97.2
150	a	(7.5589, 11.3575)	3.7985	97.5	(8.4249, 8.5884)	0.1635	97.8
	b	(1.9917, 12.9882)	10.9964	97.5	(9.6601, 10.1616)	0.5015	97.2
	λ	(1.6322, 12.4651)	10.8329	97.5	(9.3246, 9.6627)	0.3381	97.4
200	a	(7.9302, 11.2356)	3.3054	97.5	(8.4341, 8.5586)	0.1244	97.0
	b	(2.1114, 12.6055)	10.4941	97.5	(9.8299, 10.2081)	0.3782	98.6
	λ	(1.6939, 11.6296)	9.9356	97.5	(9.3760, 9.6133)	0.2372	96.9
250	a	(7.5255, 11.0986)	3.5731	97.5	(8.4439, 8.5659)	0.1220	98.1
	b	(1.8434, 12.1621)	10.3187	97.5	(9.8278, 10.2069)	0.3792	98.3
	λ	(1.6123, 11.1655)	9.5532	97.5	(9.3742, 9.6182)	0.2439	96.9
300	a	(7.6347, 10.9165)	3.2818	97.5	(8.4391, 8.5635)	0.1244	98.0
	b	(1.8615, 11.8142)	9.9526	97.5	(9.8016, 10.0174)	0.3732	97.2
	λ	(1.6127, 11.0458)	9.4331	97.5	(9.3957, 9.6259)	0.2303	98.0
350	a	(7.5348, 10.9013)	3.3665	97.5	(8.4412, 8.5639)	0.1227	98.1
	b	(1.7445, 11.5605)	9.8160	97.5	(9.8214, 10.1937)	0.3722	98.0
	λ	(1.5542, 11.2813)	9.7271	97.5	(9.3647, 9.6239)	0.2593	97.1

Table 5: Estimate Values and MSEs (With a = 11, b = 12, λ = 10).

n	Parameters	MLE			Bayesian		
		Parameters estimates	MSE	Bias	Bayes	MSE	Bias
100	a	11.9791	2.1747	9.1246	11.0016	0.0019	0.0016
	b	10.0718	11.0059	1.9526	11.9273	0.0229	-0.0727
	λ	9.5917	4.1726	1.8113	9.9985	0.0072	-0.0015
150	a	12.0203	1.9536	1.0203	10.9987	0.0010	-0.0013
	b	10.0446	9.7777	-1.9554	12.0027	0.0091	0.0027
	λ	9.6653	3.6611	-0.3347	10.0007	0.0036	0.0006
200	a	11.9769	1.6665	0.9769	10.9996	0.0011	-0.0004
	b	9.8902	8.7887	-2.1098	11.9990	0.0092	-0.0009
	λ	9.73425	3.0675	-0.2657	10.0011	0.0038	0.0011
250	a	1.5204	12.0164	1.0164	11.1001	0.0010	0.0007
	b	6.8925	10.0280	-1.9719	12.0011	0.0091	0.0011
	λ	1.9663	9.8119	-0.1880	9.9982	0.0041	-0.0017
300	a	11.9664	1.4267	0.9664	10.9987	0.0010	0.0013
	b	9.9036	7.2334	-2.0964	11.9945	0.0091	-0.0055
	λ	9.7793	2.0990	-0.2207	9.9985	0.0039	-0.0015
350	a	11.9776	1.4680	0.9776	10.9991	0.0010	-0.0009
	b	9.9074	7.7049	-2.0926	12.0077	0.0099	0.0077
	λ	9.7131	2.6663	-0.2869	9.9996	0.0040	-0.0004

Table 6: CI, HPD interval, AILs and CPs (With a = 11, b = 12, λ = 10).

n	Parameters	MLE			Bayesian		
		CI	AIL	CP	HPD Interval	AIL	CP
100	a	(9.1246, 13.9673)	4.8426	97.5	(10.9179, 11.0839)	0.1660	97.4
	b	(1.9526, 14.8705)	12.9179	97.5	(11.6609, 12.1882)	0.5273	97.8
	λ	(1.8113, 11.6084)	9.7970	97.5	(9.8345, 10.1683)	0.3338	97.4
150	a	(9.4558, 13.7115)	4.2557	97.5	(10.9307, 11.0554)	0.1247	96.8
	b	(1.9606, 14.3851)	12.4245	97.5	(11.8327, 12.2011)	0.3685	98.7
	λ	(1.8304, 11.7352)	9.9048	97.5	(9.8853, 10.0114)	0.0229	96.8
200	a	(9.7569, 13.4485)	3.6915	97.5	(10.9331, 11.0583)	0.1251	96.8
	b	(2.3320, 13.0512)	10.7191	97.5	(11.8153, 12.1889)	0.3736	97.7
	λ	(1.9158, 11.9345)	10.0187	97.5	(9.8841, 10.1201)	0.2359	97.7
250	a	(10.2618, 13.3555)	3.0937	97.5	(10.0941, 11.1066)	0.1223	98.3
	b	(3.2123, 12.7593)	9.5469	97.5	(11.8131, 12.1761)	0.3629	97.5
	λ	(2.1644, 10.4936)	8.3292	97.5	(9.8685, 10.1243)	0.2558	97.1
300	a	(9.9565, 13.1054)	3.1489	97.5	(10.9384, 11.0627)	0.1243	97.7
	b	(3.1440, 12.3056)	9.1616	97.5	(11.8156, 12.1805)	0.3649	97.8
	λ	(2.1327, 10.5613)	8.4286	97.5	(9.8793, 10.1261)	0.2468	98.0
350	a	(9.7578, 13.1574)	3.3997	97.5	(10.9401, 11.0664)	0.1263	97.8
	b	(2.5721, 12.3495)	9.7774	97.5	(11.8184, 12.1988)	0.3804	98.3
	λ	(1.9917, 10.6485)	8.6568	97.5	(9.8873, 10.1293)	0.2419	98.2

24.29, 23.56, 50.85, 24.53, 13.74, 27.99, 59.27, 13.31, 41.63, 10.00, 33.62, 32.90, 27.55, 16.76, 47.00, 106.33, 21.03.

- August: 16.00, 9.52, 9.43, 53.72, 17.10, 8.52, 10.00, 15.23, 8.78, 28.97, 28.06, 18.26, 9.69, 51.43, 10.96, 13.74, 20.01, 10.00, 12.46, 10.40, 26.99, 7.72, 11.84, 18.39, 11.22, 13.10, 16.58, 12.46, 58.98, 7.11, 11.63, 8.24, 9.80, 15.51, 37.86, 30.20, 8.93, 14.29, 12.98, 12.01, 6.80.

The results obtained using the C-FE distribution are compared with the composed-Weibull exponential (C-WE), Weibull (W) and Fréchet (Fr). We consider certain discrimination criteria such as Kolmogorov-Smirnov (K-S) test statistic, Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), and corrected Akaike information criterion (CAIC). The preferred model is the one which provides the minimum values of the aforementioned statistics. Summary statistics of the data are mentioned in Table 7.

Table 7: Summary statistics of minimum flow of water during May and August at Piracicaba River in Brazil

Month	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
May	10.00	19.46	28.59	44.96	44.51	374.54
Augus	6.80	9.80	12.46	17.44	18.26	58.98

Table 8: Estimates and Standard Errors for Different Distributions during May and August at Piracicaba River in Brazil

Month	Distributions	Estimates (standard errors)		
		$\hat{a}$	$\hat{b}$	$\hat{\lambda}$
May	C-FE	1.7994 (0.7587)	21.6957 (13.4717)	5.7000 (0.0001)
	Fr	1.4807 (0.1357)	105.172 (42.6084)	
	C-WE	0.8962 (0.1294)	0.0422 (0.0271)	0.0369 (0.0177)
	W	1.1036 (0.1136)	0.0142 (0.0074)	
August	C-FE	2.3915 (0.9098)	24.943 (14.301)	5.7000 (0.0001)
	Fr	1.9327 (0.1926)	113.1832 (51.1638)	
	C-WE	1.5670 (0.6447)	0.4712 (0.2839)	8.7000 (0.0000000105)
	W	1.5763 (0.1717)	0.0091 (0.0054)	

Parameter estimation of C-FE, Fr, W and C-WE model are tabulated in Table 8. From Table 9, it will observe that C-FE distribution fits better than the chosen models. These results are confirmed from the K-S, AIC, BIC, and CAIC values, since C-FE distribution has the minimum values for the proposed data sets. Therefore, the proposed methodology can be used successfully to analyze the minimum flow of water during May

Table 9: Analytical results for different probability distributions for the data sets related to the minimum flows of water during May and August at Piracicaba River in Brazil.

Month	Measures	C-FE	Fr	C-WE	W
	K_S(p-value)	0.0633 (0.9972)	0.0964 (0.8511)	0.13454 (0.4639)	0.1875 (0.1202)
	-2log L	354.0966	357.4268	370.6919	383.5337
May	AIC	360.0966	361.4268	376.6919	387.5337
	BIC	358.9028	360.6309	375.4981	386.7378
	CAIC	361.9028	362.6309	378.4981	388.7378
	K_S(p-value)	0.0754 (0.9739)	0.1149 (0.6514)	0.1841 (0.1242)	0.1868 (0.1144)
	-2log L	276.6462	280.4167	302.9114	302.9043
August	AIC	282.6462	284.4167	308.9114	306.9043
	BIC	281.4845	283.6422	307.7497	306.1299
	CAIC	384.4845			

and August at Piracicaba River using the C-FE distribution. The plots of the fitted C-FE, Fr, W and C-WE densities are shown in Figure 7 and 8.

Figure 7 illustrates the fitted pdfs of the C-FE, Fr, W and C-WE models for minimum monthly flows of water (m^3/s) during May.

Figure 8 illustrates the fitted pdfs of the C-FE, Fr, W and C-WE models for minimum monthly flows of water (m^3/s) during August on the Piracicaba River,

8.2 Active Repair Times for an Airborne Communication Transceiver

We provide data analysis to assess the goodness of fit of the C-FE model with respect to the active repair times (hours) for an airborne communication transceiver to see how C-EF distribution works in practice. The data is given by: 0.50,0.60,0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.5, and their source is Jorgensen (2012).

We fit a proposed model (C-FE) in comparison with six other well-known competing distributions. The cumulative functions of the competing models are:

I. Exponential (Ex) distribution

$$F(x) = 1 - e^{-\lambda x}, \qquad x, \lambda \geq 0.$$

II. Exponentiated Fréchet (EFr) distribution introduced by da Silva et al. (2013).

$$F(x) = 1 - \left(1 - \exp^{-\left(\frac{b}{x}\right)^a}\right)^{\alpha}, \quad x, \alpha, a, b \geq 0.$$

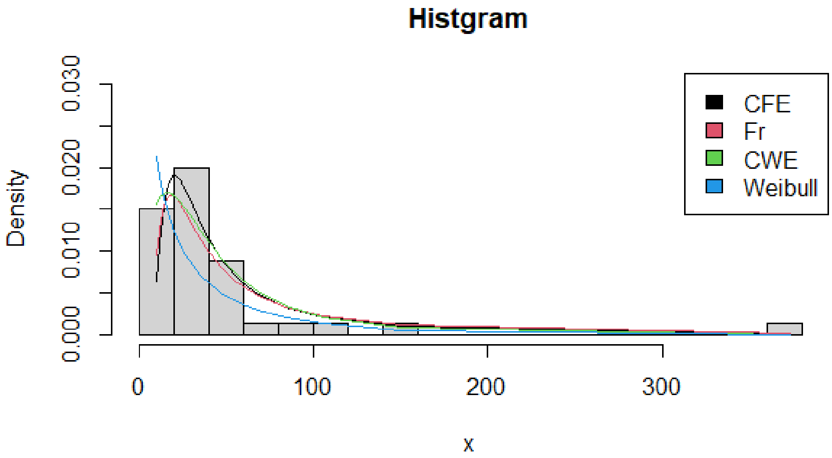


Figure 7: illustrates the fitted pdfs of the C-FE, Fr, W and C-WE models for minimum monthly flows of water (m^3/s) during May on the Piracicaba River.

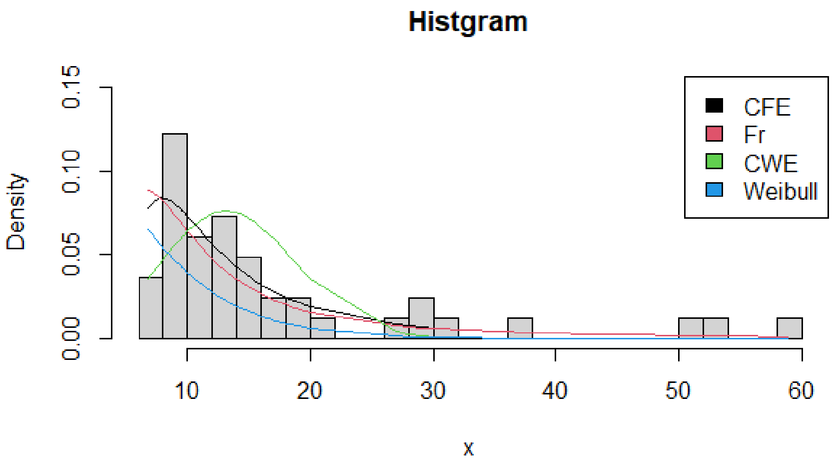


Figure 8: illustrates the fitted pdfs of the C-FE, Fr, W and C-WE models for minimum monthly flows of water (m^3/s) during August on the Piracicaba River.

III. Exponentiated Generalized Fréchet (EGFr) distribution introduced by Abd-Elfattah et al. (2016)

$$F(x) = \left[1 - \left(1 - \exp^{-\left(\frac{b}{x}\right)^a}\right)^\alpha\right]^\beta, \quad x, \ \alpha, \beta, a, \ b \geq 0 \ .$$

IV. Transmuted Exponentiated Fréchet (TEFr) distribution introduced by Elbatal et al. (2014)

$$F(x) = \left[1 - \left(1 - e^{-\left(\frac{b}{x}\right)^a}\right)^\beta\right]\left\{1 + \alpha\left[\left(1 - e^{-\left(\frac{b}{x}\right)^a}\right)\right]^\beta\right\}, \ x, \ \beta, a, \ b \geq 0, \ and \ |\alpha| \leq 1.$$

V. Topp Leone Fréchet (TLF) distribution introduced by Sapkota (2021)

$$F(x) = \left[1 - \left(1 - e^{-\left(\frac{b}{x}\right)^a}\right)^2\right]^\alpha, \ x, \ \alpha, a, \ b \geq 0.$$

VI. Alpha Power Transformed Fréchet (APTF) distribution introduced by Elbatal et al. (2018)

$$F(x) = \frac{e^{-\left(\frac{b}{x}\right)^a} \ \alpha^{e^{-\left(\frac{b}{x}\right)^a}}}{\alpha}, \quad x, \ \alpha, a, \ b \geq 0 \ and \ \alpha \neq 1.$$

The analytical measure such as Kolmogorov-Smirnov (KS) test statistic, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and corrected Akaike information criterion (CAIC) are considered for deciding the goodness of the fit results of the proposed model and other competing models. On considering these measures, it is shown that the newly proposed model provides greater distributional flexibility than the other well-known distributions. Summary statistics of the data are mentioned in Table 10.

Table 10: Summary statistics of active repair times (hours).

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.500	1.000	2.100	4.013	4.775	24.500

In Table 12, we compare the C-FE model with the Exponential (Ex), Exponentiated Fréchet (EFr), Exponentiated Generalized Fréchet (EGFr), Transmuted Exponentiated Fréchet (TEFr), the Topp Leone Fréchet (TLF),and Alpha power Transformed Fréchet (APTF) distributions. It is noted that the C-FE distribution has the lowest values for the AIC, BIC and CAIC statistics among all fitted models. So, the C-FE model can be chosen as the best model among all fitted models for this data.

Figure 9 illustrates the fitted pdfs of the C-FE, Ex, EFr, EGFr, TEFr, APTF and TLF models for active repair times (hours) data.

Table 11: Estimates of the parameters and standard errors (in parentheses) for the models fitted to active repair times (hours).

Model	**Estimates (standard errors)**				
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
C-FE	1.135 (0.172)	1.442 (0.289)	3.374 (2.187)	-	-
Ex	-	-	0.249 (0.039)	-	-
EFr	1.241 (0.972)	1.405 (1.313)	-	0.956 (1.218)	-
EGFr	6.591 (7.319)	0.387 (0.198)	-	0.156 (0.185)	3.085 (3.041)
TEFr	3.309 (21.947)	0.648 (1.767)	-	-0.535 (2.622)	0.289 (2.103)
TLF	0.665 (0.158)	0.215 (0.926)	-	16.584 (76.313)	-
APTF	1.256 (0.211)	1.321 (0.448)	-	1.246 (0.954)	-

Table 12: Analytical results of C-FE and other competing model Data

The model	Measures				
	K_S (p-value)	-2log L	AIC	BIC	CAIC
C-FE	0.091 (0.892)	177.819	183.819	188.886	184.486
Ex	0.138 (0.430)	191.153	193.153	194.842	193.258
EFr	0.096 (0.857)	178.897	184.897	189.964	185.564
EGFr	0.095 (0.865)	177.949	185.949	192.704	186.792
TEFr	0.099 (0.831)	178.638	186.638	193.393	187.881
TLF	0.096 (0.856)	179.057	185.057	190.124	185.724
APTF	0.865 (0.095)	178.806	184.806	189.873	185.473

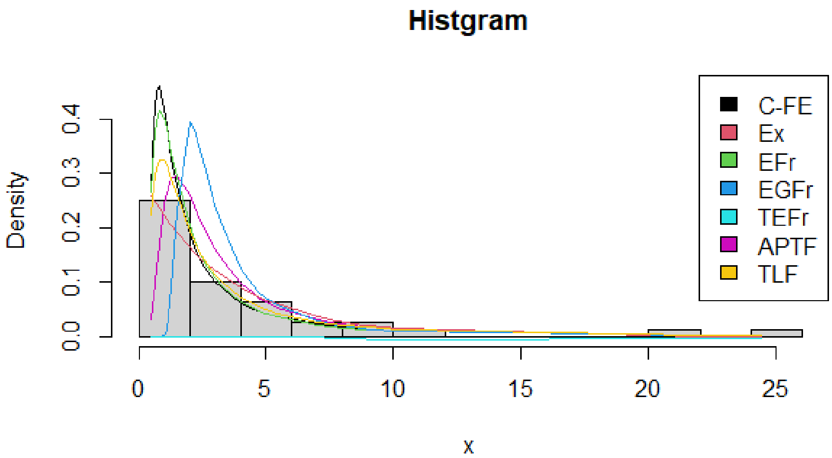


Figure 9: The fitted pdfs of the C-FE, Ex, EFr, EGFr, TEFr, APTF and TLF models for active repair times (hours) data.

9 Conclusions

We introduced a new generator based on the star-shaped property. The new class is named as the composed –G Q class. To examine the performance of the new generator and to contribute to the extreme value distributions, the new family depends on Fréchet distribution, this family called composed- Fréchet Q family. A sub-model of the new family called the composed- Fréchet exponential (C-FE) distribution is presented to provide the flexibality of the family. The statistical properties were discussed. The parameters of C-FE distribution are estimated by using the maximum likelihood and Bayesian methods. A simulation study was conducted to compare the performances of Bayes estimators with corresponding maximum likelihood estimators. We use two real sets of data to prove empirically the importance and flexibility of the new model and comparing it with other models. In conclusion the Bayesian method is the best method to obtain estimates for this new family of distributions and the new generator provides the best fit to all of the data.

Future Work

The present work can be extended

1. Estimation parameters of the composed-Fréchet exponential distribution by using different methods of estimation.
2. Estimation of C-FE distribution under different censoring samples can be performed.

3. Generating many distributions by using Composed-Fréchet Generated Family.

Appendix

Proof 1

Moments

Start with

$$E\left(X^r\right)=\int_0^{\infty} x^r f(x) \quad dx$$

and substitute for $f(x)$ from Eq.(8) sets,

$E\left(X^r\right)=\int_0^{\infty} x^r ab\left(x(1-e^{-\lambda x})\right)^{-a-1} e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}}.\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\}dx$

Since,

$e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left[b\left(x(1-e^{-\lambda x})\right)^{-a}\right]^i.$

Then,

$E\left(X^r\right)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}ab^{i+1}\int_0^{\infty} x^{r-a(i+1)-1}\left((1-e^{-\lambda x_i})\right)^{-a(i+1)-1}.\{x.\lambda e^{-\lambda x_i}+1-e^{-\lambda x_i}\}dx$

and since,

$\left(1-e^{-\lambda x_i}\right)^{-a(i+1)-1}=\sum_{k=0}^{\infty}(-1)^k\binom{-a(i+1)-1}{k}e^{-k\lambda x}.$

Then,

$$E\left(X^r\right)=\sum_{i=k=0}^{\infty}(-1)^{i+k}ab^{i+1}\int_0^{\infty} x^{r-a(i+1)-1}\binom{-a(i+1)-1}{k}e^{-k\lambda x}.\cdot\{x.\lambda e^{-\lambda x_i}+1-e^{-\lambda x_i}\}dx$$

After some simple Algebric steps, we get the result in Eq. (12).

Proof 2

Order Statistics

$$f_{i;n}(x)=\frac{n!}{(i-1)!(n-i)!}\left(F(x)\right)^{i-1}\left(1-F(x)\right)^{n-i}f(x)$$

Where $x=x_{(i)} for\ simplicity$

$$f_{i;n}(x) = \frac{n!}{(i-1)!(n-i)!}\left(e^{-b\left(x\left(1-e^{-\lambda x}\right)\right)^{-a}}\right)^{i-1}\left(1-e^{-b\left(x\left(1-e^{-\lambda x}\right)\right)^{-a}}\right)^{n-i}$$
$$ab\left(x(1-e^{-\lambda x})\right)^{-a-1}e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}}$$

$.\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\}$

Since,

$\left(1-e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}}\right)^{n-i} = \sum_{j=0}^{\infty}(-1)^j\begin{pmatrix}n-i\\ j\end{pmatrix}e^{-jb\left(x(1-e^{-\lambda x})\right)^{-a}}.$

Then,

$$f_{i;n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{\infty}(-1)^j\begin{pmatrix}n-i\\ j\end{pmatrix}ab\left(x(1-e^{-\lambda x})\right)^{-a-1}\cdot$$
$$\cdot e^{-(i+j)b\left(x(1-e^{-\lambda x})\right)^{-a}}\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\}$$

and since,

$e^{-(i+j)b\left(x(1-e^{-\lambda x})\right)^{-a}} = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left[(i+j)b\left(x(1-e^{-\lambda x})\right)^{-a}\right]^r.$

Then,

$$f_{i;n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+r}}{r!}\begin{pmatrix}n-i\\ j\end{pmatrix}(i+j)^r ab^{r+1}\left(x(1-e^{-\lambda x})\right)^{-ar}\cdot$$
$$\cdot\left(x(1-e^{-\lambda x})\right)^{-a-1}.$$

$.\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\}.$

After some simple Algebric steps, we get the result in Eq. (14).

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