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By Boulkeroua, Al-Jarallah, Raqab

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Statistical analysis of Gompertz distribution based on progressively type-II censored competing risk model with binomial removals

Fouzia B. Boulkeroua[a], Reem A. Al-Jarallah[b], and Mohammad Z. Raqab[*,a,b]

[a] *Department of Mathematics, The University of Jordan, Amman 11942, Jordan,*
[b] *Department of Statistics and OR, Kuwait University, Al-Shadadiyah, Kuwait*

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Here in this paper, we consider the progressive Type-II censoring Gompertz data under competing risks model with binomial removals. The maximum likelihood estimators of the model parameters involved are obtained by applying numerical methods and the asymptotic variance-covariance matrix of the estimates is also derived. Bayesian estimates based on importance sampling procedure are developed under squared error, LINEX and general entropy loss functions. The confidence intervals using the asymptotic normality and Bayesian approaches are also developed. Finally, a Monte Carlo simulation is conducted to evaluate the performance of the so proposed estimators under all these different estimation methods.

keywords: Gompertz distribution, Progressive Type-II censoring competing risks, Binomial removals, Importance sampling, Maximum likelihood estimation, Bayesian estimation.

1 Introduction

Gompertz (GO) distribution was first introduced by Gompertz (1825) as a generalization of exponential distribution. Since that time Gompertz distribution was applied in many areas such as epidemiological and biomedical studies, and widely used as a growth

*Corresponding author: mraqab@ju.edu.jo

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model in actuarial, life testing and reliability experiments. Variety of applications were introduced by many authors, for example, Ahuja and Nash (1967) and Pollard and Valkovics (1992). The area of censoring arises in life-testing and reliability experiments when some surviving units are removed from the experiment following a specific censoring scheme obtaining the censored sample. This situation occurs in industrial and medical applications due to many reasons such as shortage of time, funds and inadequate sources of materials. Several types of censoring schemes were discussed in the literature, and one of the most popular is Type-II censoring. It occurs when a total of n items are tested, the experiment is terminated when the k-th ($k \geq n$) item fails, instead of continuing experiment until all n items have failed. For more about censoring schemes, see the monograph Balakrishnan and Aggarwala (2000).

A progressive Type-II censoring is proposed as a generalization of Type-II censoring, that allows removing the surviving units at different stages of the test. Statistical inference based on progressive Type-II censoring data was considered by many authors, for example, Ng (2005) considered the inference of unknown parameters of a modified Weibull distribution in the presence of progressive censoring. Pradhan and Kundu (2009) studied the estimation of the generalized exponential distribution based on a progressive censored data. Raqab et al. (2010) dealt with different predictors of failure times of censored units in a progressively censored sample from Pareto distribution. Recently, Valiollahi et al. (2018) presented estimation and prediction for power Lindley distribution under progressively Type-II censored sample data. Moreover, Maurya et al. (2019) focused on inference of inverted exponential Rayleigh distribution under progressive first-failure censoring. More inference procedures and mathematical results under progressive Type-II data are given in Aggarwala and Balakrishnan (1996), Balakrishnan and Sandhu (1996), and Aggarwala and Balakrishnan (1998). For further reviews, and different developments in this area see the monographs of Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014).

In comparative studies involving more than one population, the need arises for joint progressive censoring (JPC) schemes. To illustrate that, suppose we have two independent samples of sizes m and n which are combined and placed on a life testing experiment, selected from two different groups of production, say, Group 1 and Group 2. Let $N = m+n$ be the size of the combined sample, with k, number of failures where $k \geq N$. Besides, let $R_1, R_2, \ldots, R_k$ be non-negative integers satisfying $(\sum_{i=1}^{k} R_i + k = m+n)$, where $R_i = S_i + W_i$ with S_i and W_i being the number of removals of the i-th stage from Group 1 and Group 2, respectively. At the time of the first failure, say T_1, we randomly remove $R_1 = S_1 + W_1$ units from the remaining $(N-1)$ surviving units where S_1 and W_1 are the number of removed units from Group 1 and Group 2, respectively. Continuing this process, at the time of second failure, T_2, we randomly remove $R_2 = S_2 + W_2$ items from the remaining combined $N-2-R_1$ surviving units. The experiment continues until the final stage, that is the time of k-th failure, where all the remaining $(R_k = N - k - \sum_{i=1}^{k} R_i)$ surviving units are removed. It should be noticed that the JPC scheme includes the complete sample as a special case when $R_1 = R_2 = \ldots = R_k = 0$ for all $i = 1, 2, \ldots, k$

and Type-II censoring schemes when $R_1 = R_2 = \ldots = R_{k-1} = 0$ so that $R_k = N - k$.

Extending the JPC scheme for more than two exponential populations was discussed by Balakrishnan et al. (2015). Under JPC scheme, Mondal and Kundu (2019) presented point and interval estimation of the parameters of Weibull populations, and Parsi et al. (2010) focused on the conditional maximum likelihood and interval estimation of two Weibull distributions. Doostparast et al. (2013) considered Bayes estimates for general class of distributions under JPC scheme with respect to different loss functions such as the squared error and LINEX loss functions. Moreover, Parsi and Bairamov (2009) determined the expected number of failures in life testing experiment under the JPC scheme for different families.

In survival analysis studies, interest arises to estimate a specific risk in the presence of other risk factors. This situation is very common when there are more than one cause of failure occurs at the same time. This is called competing risks analysis, and has attracted the interest in many areas, for example, engineering, social sciences, biology and medical statistics. In a lifetime experiment, we assume that we have n identical units with k being the number of known causes of failure. Their lifetimes $(T_1, T_2, \ldots, T_n)$ are assumed to be independent and identically distributed (iid) random variables where T_{kj} is the latent failure time of the j-th units under the k-th failure mode, which are usually assumed to be iid. In the progressive Type-II censoring process with competing risk, we end up with a sample consisting of a time to failure and the cause of failure, that is $(\boldsymbol{T}, \boldsymbol{\delta}, \boldsymbol{R}) = (T_1, \delta_1, R_1), (T_2, \delta_2, R_2), \ldots, (T_m, \delta_m, R_m)$, where $T_1 < T_2 < \ldots < T_m$ denote the $m(m \leq n)$ observed failure times, $\delta_1, \ldots, \delta_m$ denote the causes of failure, and $R_1, \ldots, R_m$ be the number of units removed from the study at the failure time $T_1, T_2, \ldots, T_m$ satisfying $R_1+R_2+\ldots+R_m+m = n$. In this paper, we assume that there are only two independent causes of failure (*i.e.* $k = 2$). Kundu et al. (2003) presented inferential methods for competing risk data under progressive Type-II censoring. Pareek et al. (2009) studied the same model under the assumption of independent Weibull distributed latent failure times. Cramer and Schmiedt (2011) investigated competing risks data from the Lomax distribution based on progressive Type-II censoring, while the competing risks model using half-logistic distribution was introduced by AL-Hussaini et al. (2015). Mao et al. (2014) considered a statistical inference from exponential competing risks model under different censoring systems. In the Bayesian setting, Hemmati and Khorram (2011) dealt with Bayesian analysis of the adaptive Type-II progressively hybrid censoring scheme. Also Dey et al. (2016) considered Bayesian analysis of a modified Weibull distribution under a progressively censored competing risk model.

In some practical situations, the pattern of removal at each failure time may be assumed to be random. Thus, progressive censoring with random removal has attracted a considerable attention. For instance, Sarhan et al. (2008) discussed the competing risks model when the data are progressively type-II censored with random removals and exponential life times. Hashemi and Amiri (2011) dealt with Weibull distribution under progressive type-II censoring for competing risk data with binomial removals. The estimation problem of Rayleigh distribution under Type-II progressive censoring with binomial removals was discussed by Dey and Dey (2014). Also, Nie and Gui (2019) considered the competing risk model based on Lindley distribution under the progressive Type-II censored sample data with binomial removals. Most recently, Qin and Gui (2020) discussed the analysis of Burr XII distribution based on the competing risks model under the progressive Type-II censoring. For practical studies on competing risk model, one may refer to Bakoyannis and Touloumi (2011).

The rest of the paper is organized as follows: In Section 2, we introduce the model and present its respective notations used throughout the paper. Section 3 provides inference for the different unknown parameters using frequentist methods such as maximum likelihood estimation and asymptotic normality of the maximum likelihood estimators (MLEs). Section 4 discusses Bayesian estimators and corresponding credible intervals (CrIs) using different loss functions such as squared error, LINEX and general entropy loss functions by applying importance sampling procedure. A comprehensive simulation study is developed in Section 5 to assess the performance of the proposed estimates. Finally, concluding remarks are presented in Section 6.

2 Model Description and Notations

Suppose that we have n identical units with K failure modes having the lifetimes $T_1, T_2, \ldots, T_n$ which are assumed to be iid random variables. We assume that the failure modes (*i.e.* $k = 1, 2$) that are independent. Let $T_i = \min\{T_{1i}, T_{2i}\}$ for $i = 1, \ldots, n$, where $T_{ki}, k = 1, 2$ is the latent failure time of the i-th unit under k-th failure mode. The latent failure times T_{ki} are usually assumed to be iid for $i = 1, 2, \ldots, n$ following Gompertz distribution $GO(\alpha_k, \beta_k), k = 1, 2$. In the presence of the progressive Type-II censoring scheme under competing risks data, with only m failure time, we have the following samples $(T_1, \delta_1, R_1), (T_2, \delta_2, R_2), \ldots, (T_m, \delta_m, R_m)$, where $T_i(1, \ldots, m)$ are to be observed with the corresponding cause of failure $\delta_i \in (\delta_i\{1, 2\})$, with $R_1, R_2, \ldots, R_m$ being the number of units removed.

The CDF and PDF of the *k-th* failure cause are given by

$$F(t; \alpha_k, \beta_k) = 1 - \exp(-\alpha_k(e^{\beta_k t} - 1)), \qquad t \geq 0 \text{ and } \alpha_k,\ \beta_k > 0,\ (k = 1, 2), \tag{1}$$

and

$$f(t; \alpha_k, \beta_k) = \alpha_k \beta_k e^{(\beta_k t - \alpha_k(e^{\beta_k t} - 1))}, \qquad t \geq 0 \text{ and } \alpha_k,\ \beta_k > 0,\ (k = 1, 2). \tag{2}$$

From (1), the CDF of T_j is

$$\begin{aligned} F(t) &= 1 - P(T_j > t) \\ &= 1 - P(T_{1j} > t)P(T_{2j} > t) \\ &= 1 - [e^{-\alpha_1(e^{\beta_1 t} - 1)}][e^{-\alpha_2(e^{\beta_2 t} - 1)}], \qquad t,\ \alpha_k,\ \beta_k > 0, \quad k = 1, 2. \end{aligned} \tag{3}$$

Moreover, the survival function $S(t) = 1 - F(t)$, and the hazard function $h(t) = f(t)/(1 - F(t))$, can be easily obtained. Let

$$I(\delta_i = k) = \begin{cases} 1, & \delta_i = k, \\ 0, & otherwise, \end{cases} \tag{4}$$

and $m_k = \sum_{i=1}^{m} I(\delta_i = k)$ which gives the total number of units that failed as a result of cause k $(k = 1, 2)$. It is worth noting that $m_1 + m_2 = m$.

As we previously stated that the number of units removed at each failure time is random, independent and following a binomial distribution of probability p, thus we have

$$P(\boldsymbol{R} = \boldsymbol{r}) = \frac{(n-m)!}{\prod_{k=1}^{m-1} r_k! \left(n - m - \sum_{k=1}^{m-1} r_k\right)!} \, p^{\sum_{k=1}^{m-1} r_k} \, (1-p)^{(m-1)(n-m) - \sum_{k=1}^{m-1} (m-k) r_k}.$$

Table 1 presents an illustration of the model with binomial removals.

Table 1: The schematic representation of the JPC with binomial removals

Process	The number in life testing	Failures	Binomial removals	Remains
1	n	1	$R_1 \sim B(n-m, p)$	$n - 1 - R_1$
2	$n - 1 - R_1$	1	$R_2 \sim B(n - m - R_1, p)$	$n - 2 - R_1 - R_2$
...	...	...	...	...
$m-1$	$n - (m-1) - \sum_{k=1}^{m-2} R_k$	1	$R_{m-1} \sim B(n - m - \sum_{k=1}^{m-2} R_k, p)$	$n - (m-1) - \sum_{k=1}^{m-1} R_k$
m	$n - m - \sum_{k=1}^{n-1} R_k$	1	$R_m = n - m - \sum_{k=1}^{m-1} R_k$	0

3 Maximum Likelihood Estimation

For a given censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$ and for $k = 1, 2$, the likelihood function of $\boldsymbol{\phi} = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ based the observed data $(t_1, \delta_1, r_1), \ldots, (t_m, \delta_m, r_m)$ is given by

$$\begin{aligned} L(\boldsymbol{\phi}; \boldsymbol{t} | \boldsymbol{R} = \boldsymbol{r}) &= A \, \alpha_1^{m_1} \, \beta_1^{m_1} \, \alpha_2^{m_2} \, \beta_2^{m_2} \, e^{\sum_{i=1}^{m} \sum_{k=1}^{2} \beta_k t_i I(\delta_i = k)} \, e^{-\sum_{i=1}^{m} \sum_{k=1}^{2} \alpha_k \, (e^{\beta_k t_i} - 1) I(\delta_i = k)} \\ &\times \; e^{-\sum_{i=1}^{m} \sum_{k=1}^{2} \alpha_k (e^{\beta_k t_i} - 1)[I(\delta_i = 3-k) + r_i]}, \end{aligned} \tag{5}$$

where $A = n(n - r_1 - 1)...(n - r_1 - ... - r_{m-1} - m + 1)$ is the normalizing constant.

Furthermore, assuming T_i and R_i are independent, the joint likelihood function have the following form:

$$L(\boldsymbol{\phi}; \boldsymbol{t}; \boldsymbol{R} = \boldsymbol{r}) = A^* \, L_1(\boldsymbol{\phi}) \, L_2(p), \tag{6}$$

where

$$\begin{aligned} L_1(\boldsymbol{\phi}) &= \alpha_1^{m_1} \, \beta_1^{m_1} \, \alpha_2^{m_2} \, \beta_2^{m_2} \, e^{\sum_{i=1}^{m} \sum_{k=1}^{2} \beta_k t_i I(\delta_i = k)} \, e^{-\sum_{i=1}^{m} \sum_{k=1}^{2} \alpha_k \, (e^{\beta_k t_i} - 1) I(\delta_i = k)} \\ &\times \; e^{-\sum_{i=1}^{m} \sum_{k=1}^{2} \alpha_k (e^{\beta_k t_i} - 1)[I(\delta_i = 3-k) + r_i]}, \end{aligned} \tag{7}$$

and

$$L_2(p) = p^{\sum_{k=1}^{m-1} r_k} (1-p)^{(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k)r_k},$$

with

$$A^* = \frac{A\ (n-m)!}{\prod_{k=1}^{m-1} r_k!(n-m-\sum_{k=1}^{m-1} r_k)!}.$$

Taking the logarithm of the likelihood function in (6) and ignoring the additive constant, the log-likelihood function of L_1 can be written as

$$\begin{aligned} l_1(\boldsymbol{\phi}) &= \log\ L_1(\boldsymbol{\phi}) = \sum_{k=1}^{2} m_k \log \alpha_k + \sum_{k=1}^{2} m_k \log \beta_k + \sum_{i=1}^{m}\sum_{k=1}^{2} \beta_k\ t_i\ I(\delta_i = k) \\ &- \sum_{i=1}^{m}\sum_{k=1}^{2} \alpha_k (e^{\beta_k t_i} - 1)(r_i + 1). \end{aligned} \tag{8}$$

The MLEs of α_k and β_k can be obtained by differentiating (8) with respect to α_k and β_k for $k = 1, 2$, and equating to zeros, that is

$$\frac{\partial l}{\partial \alpha_k} = \frac{m_k}{\alpha_k} - \sum_{i=1}^{m} (r_i + 1)\ e^{\beta_k t_i} + n = 0, \tag{9}$$

$$\frac{\partial l}{\partial \beta_k} = \frac{m_k}{\beta_k} + \sum_{i=1}^{m} t_i\ I(\delta_i = k) - \sum_{i=1}^{m} \alpha_k t_i e^{\beta_k t_i} (r_i + 1). \tag{10}$$

From (9), we get

$$\hat{\alpha_k} = \frac{m_k}{\sum_{i=1}^{m} (r_i + 1)\ e^{\beta_k t_i} - n}. \tag{11}$$

By substituting (11) into (10), we immediately obtain

$$\frac{\partial l}{\partial \beta_k} = \frac{m_k}{\beta_k} + \sum_{i=1}^{m} t_i\ I(\delta_i = k) - \frac{m_k \sum_{i=1}^{m} (r_i + 1)\ t_i\ e^{\beta_k t_i}}{\sum_{i=1}^{m} (r_i + 1)\ e^{\beta_k t_i} - n} = 0. \tag{12}$$

Theorem 1: For $m_1 > 0$ and $m_2 > 0$, the MLEs of the all parameters α_1, α_2, β_1, and β_2 exist and unique.

Proof: To show that MLEs exist and unique, we have

1. For given $\beta_k > 0$, and from Eq. (9), we have
   a) $\frac{\partial^2 l}{\partial \alpha_k^2} = -\frac{m_k}{\alpha_k^2} < 0,$

b) $\lim_{\alpha_k \longrightarrow 0^+} \frac{\partial l}{\partial \alpha_k} = +\infty$,

c) $\lim_{\alpha_k \longrightarrow +\infty} \frac{\partial l}{\partial \alpha_k} < 0$.

Therefore, $\frac{\partial l}{\partial \alpha_k}$ is a decreasing function starting from $+\infty$ to a negative constant. Consequently, there is a unique solution for $\frac{\partial l}{\partial \alpha_k} = 0$. This shows the existence and uniqueness for the MLEs of α_k, $k = 1, 2$.

2. For given $\alpha_k > 0$, and from Eq. (10), we have

a) $\frac{\partial^2 l}{\partial \beta_k^2} = -\frac{m_k}{\beta_k^2} - \alpha_k \sum_{i=1}^{m} (r_i + 1)\, t_i^2\, e^{\beta_k t_i} < 0$,

b) $\lim_{\beta_k \longrightarrow 0^+} \frac{\partial l}{\partial \beta_k} = +\infty$,

c) $\lim_{\beta_k \longrightarrow +\infty} \frac{\partial l}{\partial \beta_k} = -\infty$.

Therefore, $\frac{\partial l}{\partial \beta_k}$ is a decreasing function starting from positive $+\infty$ to negative ∞, and a unique solution for $\frac{\partial l}{\partial \beta_k} = 0$ exists. This implies that the MLEs of $\beta_k (k = 1, 2)$ are unique.

Clearly that the MLEs of $\alpha_1, \alpha_2, \beta_1, \beta_2$ can be derived by solving the non-linear equations $\frac{\partial l}{\partial \alpha_k} = 0$ and $\frac{\partial l}{\partial \beta_k} = 0$, $k = 1, 2$ by applying numerical techniques such as Newton-Raphson methods. This shows Theorem 1. Moreover, the MLE of p, can be obtained by maximizing $l_2(p)$, the log-likelihood of L_2 given by

$$l_2(p) = \left(\sum_{k=1}^{m-1} r_k \right) \log p + \left((m-1)(n-m) - \sum_{k=1}^{m-1} (m-k) r_k \right) \log(1-p).$$

Thus, the MLE of p is derived to be

$$\hat{p} = \frac{\sum_{k=1}^{m-1} r_k}{(m-1)(n-m) - \sum_{k=1}^{m-1} (m-k-1) r_k}. \tag{13}$$

3.1 CIs of Model Parameters

The CIs of the model parameters $\boldsymbol{\phi}^* = (\alpha_1, \alpha_2, \beta_1, \beta_2, p)$ are constructed by applying the asymptotic distribution theory of MLEs. The MLEs $\hat{\boldsymbol{\phi}}^*$ of $\boldsymbol{\phi}^*$ can be described as $(\hat{\boldsymbol{\phi}}^* - \boldsymbol{\phi}^*) \xrightarrow{D} N_5(0, I^{-1}(\boldsymbol{\phi}^*))$, where $I^{-1}(\boldsymbol{\phi}^*)$ is the inverse of the observed information matrix of $\boldsymbol{\phi}^*$ that can be approximated by $I^{-1}(\hat{\boldsymbol{\phi}}^*)$. Thus we have

$$I(\hat{\boldsymbol{\phi}}^*) = \left(-\frac{\partial^2 l}{\partial \boldsymbol{\phi}^*{}_i \partial \boldsymbol{\phi}^*{}_j} \right)_{5 \times 5} \Bigg|_{\boldsymbol{\phi}^* = \hat{\boldsymbol{\phi}}^*},$$

where the elements of matrix $I(\boldsymbol{\phi}^*)$ are given as follows:

- $-\frac{\partial^2 l}{\partial\alpha_1\partial\alpha_2} = -\frac{\partial^2 l}{\partial\alpha_1\partial\alpha_1} = -\frac{\partial^2 l}{\partial\alpha_1\partial\beta_2} = -\frac{\partial^2 l}{\partial\alpha_2\partial\beta_1} = 0.$
- $-\frac{\partial^2 l}{\partial\beta_1\partial\alpha_2} = -\frac{\partial^2 l}{\partial\beta_2\partial\alpha_1} = -\frac{\partial^2 l}{\partial\beta_1\partial\beta_2} = 0.$
- $-\frac{\partial^2 l}{\partial\alpha_1\partial p} = -\frac{\partial^2 l}{\partial p\partial\alpha_1} = -\frac{\partial^2 l}{\partial p\partial\alpha_2} = 0.$
- $-\frac{\partial^2 l}{\partial\beta_1\partial p} = -\frac{\partial^2 l}{\partial\beta_2\partial p} = 0.$
- $-\frac{\partial^2 l}{\partial\alpha_k\partial\beta_k} = -\frac{\partial^2 l}{\partial\beta_k\partial\alpha_k} = \sum_{i=1}^{m}(r_i+1)\ t_i\ e^{\beta_k t_i}.$
- $-\frac{\partial^2 l}{\partial\alpha_k^2} = \frac{m_k}{\alpha_k^2}.$
- $-\frac{\partial^2 l}{\partial\beta_k^2} = \frac{m_k}{\beta_k^2} + \alpha_k \sum_{i=1}^{m}(r_i+1)\ t_i^2\ e^{\beta_k t_i}.$
- $-\frac{\partial^2 l}{\partial p^2} = \frac{\sum_{k=1}^{m-1} r_k}{p^2} - \frac{(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k-1)r_k}{(1-p)^2}.$

Therefore, the $100(1-\gamma)\%$ CI of $\boldsymbol{\phi_j^*}$ is $\left(\boldsymbol{\hat{\phi}_j^*} \pm z_{\gamma/2}\sqrt{I^{-1}(\boldsymbol{\hat{\phi}^*})_{jj}}\right)$, $j = 1, ..., 5$, where $z_\gamma/2$ is the upper $\gamma/2$ quantile of the standard normal distribution. The issue of having a negative lower bound of CI, for positive parameters, can be avoided by applying the delta principle and logarithmic transformation. The asymptotic distribution of $log\ \boldsymbol{\hat{\phi}_j^*}$ is

$$(log\ \boldsymbol{\hat{\phi}_j^*} - log\ \boldsymbol{\phi_j^*}) \xrightarrow{D} N(0, Var(log\ \boldsymbol{\hat{\phi}_j^*})),$$

where

$$Var(log\ \boldsymbol{\hat{\phi}_j^*}) = \frac{Var(\boldsymbol{\hat{\phi}_j^*})}{\boldsymbol{\hat{\phi}_j^{*2}}} = \frac{I^{-1}(\boldsymbol{\hat{\phi}_j^*})}{\boldsymbol{\hat{\phi}_j^{*2}}}.$$

Then, the $100(1-\gamma)\%$ CI of $\boldsymbol{\phi_j^*}$ can be written as

$$\left(\frac{\boldsymbol{\hat{\phi}_j^*}}{e^{z_{1-\gamma/2}\sqrt{Var(\log\ \boldsymbol{\hat{\phi}_j^*})}}}, \boldsymbol{\hat{\phi}_j^*}\ e^{z_{1-\gamma/2}\sqrt{Var(\log\ \boldsymbol{\hat{\phi}_j^*})}}\right),\ j = 1, \ldots, 5.$$

4 Bayesian Estimation

In this section, the Bayesian estimators (BEs) of the model parameters are derived under different loss functions. The error loss functions which are considered here are squared error loss function (SE), asymmetric LINEX (LN) and entropy error (EE). These loss functions are defined as

$$L_{SE} = (\theta - d)^2, \qquad L_{LN} = e^{\nu(d-\theta)} - \nu(d-\theta) - 1, \quad \nu \neq 0,$$

and

$$L_{EE} = (\frac{d}{\xi})^q - q\log(\frac{d}{\xi}) - 1, \qquad q \neq 0, \xi \neq 0,$$

respectively, where θ is any parameter estimated by a decision rule d. The constants ν and q reflect the direction and the magnitude of asymmetry. In order to implement a Bayesian analysis, prior distributions on the parameters are required. We assume that the prior distributions for the unknown parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ follow independent gamma distribution, $[G(a_k, b_k)$, for $\alpha_k, k = 1, 2$ and $\beta_k, k = 3, 4]$ and the parameter p has a beta distribution $\beta(a_5, b_5)$ with the PDFs given by:

$$\pi_k(\alpha_k) = \frac{b_k^{a_k}}{\Gamma(a_k)} \alpha_k^{a_k-1}\, e^{-b_k\alpha_k}, \qquad \alpha_k > 0,\ a_k > 0,\ b_k > 0,\ (k = 1, 2),$$

$$\pi_{k+2}(\beta_k) = \frac{b_k^{a_k}}{\Gamma(a_k)} \beta_k^{a_{k+2}-1}\, e^{-b_{k+2}\beta_k}, \qquad \beta_k > 0,\ a_{k+2} > 0,\ b_{k+2} > 0,\ (k = 1, 2),$$

and

$$\pi_5(p) = \frac{\Gamma(a_5 + b_5)}{\Gamma(a_5)\Gamma(b_5)}\, p^{a_5-1}\, (1-p)^{-b_5-1}, \qquad 0 < p < 1,\ a_5 > 0,\ b_5 > 0.$$

Based on the above priors, the joint prior PDF can be written as

$$\pi(\alpha_1, \alpha_2, \beta_1, \beta_2, p) = \pi_5(p) \times \prod_{k=1}^{2} \pi_k(\alpha_k)\, \pi_{k+2}(\beta_k), \qquad \text{for } k = 1, 2. \tag{14}$$

By combining the likelihood function in (5) and the joint prior function in (14), we readily obtain the joint posterior distribution of $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ and p given the observed data. The functional form of the posterior density can be written as

$$\begin{aligned}
\pi(\boldsymbol{\phi}|\boldsymbol{t}) &= \pi(\alpha_1, \alpha_2, \beta_1, \beta_2, p) \times L(\boldsymbol{\phi}; \boldsymbol{t}; \boldsymbol{R} = \boldsymbol{r}) \\
&\propto \prod_{k=1}^{2} \alpha_k^{m_k+a_k-1} \prod_{k=1}^{2} \beta_k^{m_k+a_{k+2}-1}\, e^{-\sum_{k=1}^{2} b_k\, \alpha_k - \sum_{k=1}^{2} b_{k+2}\, \beta_k}\, e^{\sum_{i=1}^{m}\sum_{k=1}^{2} \beta_k t_i I(\delta_i=k)} \\
&\times\ e^{-\sum_{i=1}^{m}\sum_{k=1}^{2} \alpha_k\, (e^{\beta_k t_i}-1) I(\delta_i=k)}\, e^{-\sum_{i=1}^{m}\sum_{k=1}^{2} \alpha_k (e^{\beta_k t_i}-1)[I(\delta_i=3-k)+r_i]} \\
&\times\ p^{a_5+\sum_{k=1}^{m-1} r_k - 1}\, (1-p)^{b_5+(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k)r_k - 1} \\
&\propto \alpha_1^{m_1+a_1-1}\, e^{-\alpha_1\left(b_1+\sum_{i=1}^{m} (r_i+1)\, (e^{\beta_1 t_i}-1)\right)} \times \alpha_2^{m_2+a_2-1}\, e^{-\alpha_2\left(b_2+\sum_{i=1}^{m} (r_i+1)\, (e^{\beta_2 t_i}-1)\right)} \\
&\times\ \beta_1^{m_1+a_3-1}\, e^{-b_3\, \beta_1}\, e^{\beta_1 \sum_{i=1}^{m} t_i I(\delta_i=1)} \times \beta_2^{m_2+a_4-1}\, e^{-b_4\, \beta_2}\, e^{\beta_2 \sum_{i=1}^{m} t_i I(\delta_i=2)} \\
&\times\ p^{a_5+\sum_{k=1}^{m-1} r_k - 1}\, (1-p)^{b_5+(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k)r_k - 1} \\
&\propto g_1(\alpha_1|\beta_1, \boldsymbol{t}) \times g_2(\alpha_2|\beta_2, \boldsymbol{t}) \times g_3(\beta_1|\boldsymbol{t}) \times g_4(\beta_2|\boldsymbol{t}) \times g_5(p) \times h(\beta_1, \beta_2|\boldsymbol{t}),
\end{aligned} \tag{15}$$

where $g_1(\alpha_1|\beta_1,\boldsymbol{t})$ is a PDF of $G\left(m_1+a_1,\ b_1+\sum_{i=1}^{m}(r_i+1)\,(e^{\beta_1 t_i}-1)\right)$, $g_2(\alpha_2|\beta_2,\boldsymbol{t})$ is a PDF of $G\left(m_2+a_2,\ b_2+\sum_{i=1}^{m}(r_i+1)\,(e^{\beta_2 t_i}-1)\right)$, $g_3(\beta_1|\boldsymbol{t})$ is a PDF of $G\,(m_1+a_3;b_3)$, $g_4(\beta_2|\boldsymbol{t})$ is a PDF of $G\,(m_2+a_4,b_4)$, and

$$h(\beta_1,\beta_2|\boldsymbol{t})=W_{a_1,b_1}(\boldsymbol{t},\beta_1).W_{a_2,b_2}(\boldsymbol{t},\beta_2),$$

where

$$W_{a_k,b_k}(\boldsymbol{t},\beta_k)=\frac{e^{\beta_k\sum_{i=1}^{m}t_iI(\delta_i=k)}}{\left(b_k+\sum_{i=1}^{m}(r_i+1)(e^{\beta_k\,t_i}-1)\right)^{a_k}},\quad k=1,2,$$

and $g_5(p)$ is a PDF of Beta(ξ_1,ξ_2), where

$$\xi_1=a_5+\sum_{k=1}^{m-1}r_k,\ \xi_2=b_5+(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k)r_k.$$

Based on the competing risk censored data under progressive Type-II scheme with binomial removal, the BEs of α_1 and α_2 under SE, LN and EE loss functions can be derived as

$$\hat{\alpha}_{iS}=(m_i+a_i)\left(\frac{E_{\beta_i}W_{m_i+a_i+1}(\boldsymbol{t},\beta_i)}{E_{\beta_i}W_{m_i+a_i}(\boldsymbol{t},\beta_i)}\right),\quad i=1,2,$$

where E_{β_1} and E_{β_2} are the expected values with respect to the density of β_1, and β_2 which are $G(m_1+a_3,b_3)$ and $G(m_2+a_4,b_4)$, respectively.
Under LINEX and EE loss functions, the BEs of $\alpha_i(i=1,2)$ are

$$\hat{\alpha}_{iL}=-\frac{1}{\nu}\log\left[\frac{E_{\beta_i}W_{m_i+a_i,b_i+\nu}(\boldsymbol{t},\beta_i)}{E_{\beta_i}W_{m_i+a_i,b_i}(\boldsymbol{t},\beta_i)},\right]\quad i=1,2,$$

$$\hat{\alpha}_{iE}=\left(\frac{\Gamma(m_i+a_i-q)}{\Gamma(m_i+a_i)}\right)^{-\frac{1}{q}}\left[\frac{E_{\beta_i}W_{m_i+a_i-q,b_i}(\boldsymbol{t},\beta_i)}{E_{\beta_i}W_{m_i+a_i,b_i}(\boldsymbol{t},\beta_i)}\right]^{-\frac{1}{q}},\quad i=1,2.$$

respectively. The BEs of β_1 and β_2 under SE, LN and EE loss functions are given by

$$\hat{\beta}_{iS}=\frac{E_{\beta_i}\left(\beta_i\,W_{m_i+a_i}(\boldsymbol{t},\beta_i)\right)}{E_{\beta_i}\left(W_{m_i+a_i}(\boldsymbol{t},\beta_i)\right)},\quad i=1,2,$$

$$\hat{\beta}_{iL}=-\frac{1}{\nu}\log\left[\frac{E_{\beta_i}\left(e^{-\nu\beta_i}W_{m_i+a_i,b_i}(\boldsymbol{t},\beta_i)\right)}{E_{\beta_i}W_{m_i+a_1,b_i}(\boldsymbol{t},\beta_i)}\right]\quad i=1,2,$$

and

$$\hat{\beta}_{iE}=\left[\frac{E_{\beta_i}\left(\beta_i^{-q}W_{m_i+a_i,b_i}(\boldsymbol{t},\beta_i)\right)}{E_{\beta_i}W_{m_i+a_i,b_i}(\boldsymbol{t},\beta_i)}\right]^{-\frac{1}{q}}\quad i=1,2.$$

Furthermore, the BEs of p under SE, LN and EE loss functions, are readily obtained, respectively, as

$$\hat{p}_S = \xi_1/(\xi_1 + \xi_2), \qquad \hat{p}_L = -\frac{1}{\nu}\log\left[\int_0^1 e^{-\nu p} g_5(p)\, dp\right],$$

and

$$\hat{p}_E = \left[\int_0^1 p^{-q} g_5(p)\, dp\right]^{-\frac{1}{q}}.$$

Unfortunately, all the previous BEs of the model parameters can not be obtained explicitly, thus the sample-based methods were applied to calculate these estimates.

Algorithm for obtaining sample-based estimates:

- Generate p from $g_5(p)$;
- Generate β_1 from $g_3(\beta_1|\boldsymbol{t})$ and generate β_2 from $g_4(\beta_2|\boldsymbol{t})$;
- Generate α_1 from $g_1(\alpha_1|\beta_1, \beta_2, \boldsymbol{t})$ and α_2 from $g_2(\alpha_2|\beta_2, \boldsymbol{t})$;
- Repeat steps 1-4, M times and then obtain

  $(p_1, \beta_{11}, \beta_{21}, \alpha_{11}, \alpha_{21}), \ldots, (p_M, \beta_{1M}, \beta_{2M}, \alpha_{1M}, \alpha_{2M})$;
- Compute

$$w_i(p_i, \beta_{1i}, \beta_{2i}, \alpha_{1i}, \alpha_{2i}) = \frac{h(\beta_{1i}, \beta_{2i}, \boldsymbol{t})}{\sum_{i=1}^{M} h(\beta_{1i}, \beta_{2i}, \boldsymbol{t})},$$

  and then compute an approximate BE of any function of p, α_1, α_2, β_1 and β_2 (say, $\lambda(p, \alpha_1, \alpha_2, \beta_1, \beta_2)$ under a SE, LN and EE loss function as follows:

$$\hat{\lambda}_S = \sum_{i=1}^{M} w_i(p_i, \beta_{1i}, \beta_{2i}, \alpha_{1i}, \alpha_{2i})\ \lambda(p, \alpha_1, \alpha_2, \beta_1, \beta_2),$$

$$\hat{\lambda}_L = -\frac{1}{\nu} log\left[\sum_{i=1}^{M} w_i(p_i, \beta_{1i}, \beta_{2i}, \alpha_{1i}, \alpha_{2i})\ e^{-\nu\lambda(p_i, \alpha_{1i}, \alpha_{2i}, \beta_{1i}, \beta_{2i})}\right],$$

$$\hat{\lambda}_E = \left[\sum_{i=1}^{M} w_i(p_i, \beta_{1i}, \beta_{2i}, \alpha_{1i}, \alpha_{2i})\ \lambda(p, \alpha_1, \alpha_2, \beta_1, \beta_2)^{-q}\right]^{-1/q}.$$

5 Simulation Study

In this section, a simulation study is conducted based on JPC under competing risks Gompertz (GO) model, to assess the behavior of different proposed estimators of the model parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, p$ for different combinations of sample sizes n, effective sample sizes m and removal probabilities p. All our numerical computations are performed using R software (R Development Core Team , 2018). For estimating a parameter θ by $\hat{\theta}$, we define the following optimality criteria:
(1) the bias of $\hat{\theta}$, $Bias(\hat{\theta}) = E(\hat{\theta} - \theta)$;
(2) the means square error (MSE) of $\hat{\theta}$ is $MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2$;
(3) the average length (AL) of a confidence interval (L, U) is $E(U - L)$;
(4) the coverage probability (CP) of a confidence interval (L, U) is the percentage of the true value lying in this confidence interval.

The performance of the MLEs and BEs of the model parameters has been compared in terms of average bias and MSE. The CIs and CrIs are assessed in terms of the AL and CP criteria. Following the algorithm proposed by Balakrishnan and Aggarwala (2000), we generate progressive censoring data using competing risks model and random removals from GO distribution for competing failure mode j. First, we generate the progressive censoring scheme $r_1, r_2, \ldots, r_k$, with $r_i \sim B(n - r - \sum_{j=1}^{i-1} r_j, p)$, with the removal probability p. Secondly, we generate progressive censoring data sample with random removals from GO distribution for competing failure mode j. Here, we consider two competing failure modes ($j = 1, 2$), with initial values of parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (0.7, 0.8, 0.2, 0.3)$. Table 2 presents different progressive schemes that are used for different choices of n, m and p. For a specific progressive scheme, we use the following notations, $k = 5$ and $R = (3, 0_{(4)})$ denotes $R_1 = 3, R_2 = R_3 = R_4 = 0$.

The MLEs are computed by maximizing the likelihood function and solving the likelihood equations in (11) and (12). For the Bayesian estimation, two different priors were considered. The first prior (Prior 0), is known as the non-informative prior, with $a_1 = b_1 = a_2 = b_2 = 0$, while the second one is informative prior (Prior 1) with $a_1 = a_2 = a_3 = 1, b_1 = b_2 = b_3 = 2$. Furthermore, the BEs are computed using different values of ν and q, $(\nu = -0.5, 0.5, q = -0.5, 0.5)$, respectively. Tables 3 -10 present the average biases, MSEs, ALs and CPs based on MLEs and BEs of the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, p)$, obtained from 10000 replications, under SE, LN and EE loss functions with Prior 0 and Prior 1.

It can be seen that for fixed p and n, the MSEs, bias and ALs decrease as the effective sample size m increases. Furthermore as p increases, these quantities are decreased. Moreover, it is observed that the BEs of $\alpha_1, \alpha_2, \beta_1$ and β_2 based on symmetric and asymmetric loss functions are all performing well. It is clear to note that the BEs under different loss functions perform better than the MLEs for all considered cases of censoring schemes. Moreover, the results of the BEs under SE, LN and EE are close. For the interval estimation, the ALs of the BEs are lower than that of the MLEs. By considering the CP criterion, the BEs are the most valid procedure since the CPs are about the true level.

Table 2: The censoring schemes used

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Index	n	m	p	R	notation
1	30	5	0.09	$(4,6,5,6,4)$	R_1^*
2			0.4	$(2,13,6,2,2)$	R_2^*
3			0.8	$(3,20,2,0,0)$	R_3^*
4		10	0.09	$(0,1,2,1,4,4,2,0,4,2)$	R_4^*
5			0.4	$(4,5,4,4,2,1,0,0,0,0)$	R_5^*
6			0.8	$(10,7,3,\underbrace{0}_{7\ times})$	R_6^*
7		15	0.09	$(1,3,2,3,0,0,3,2,0,0,1,0,0,0,0)$	R_7^*
8			0.4	$(5,5,3,0,1,0,0,0,0,1,0,0,0,0,0)$	R_8^*
9			0.8	$(14,1,\underbrace{0}_{13\ times})$	R_9^*
10	40	10	0.09	$(2,5,4,4,2,5,3,1,1,3)$	R_{10}^*
11			0.4	$(1,12,5,5,3,1,2,1,0,0)$	R_{11}^*
12			0.8	$(8,16,5,1,\underbrace{0}_{6\ times})$	R_{12}^*
13		15	0.09	$(0,4,1,2,3,2,4,1,0,2,2,1,1,2,0)$	R_{13}^*
14			0.4	$(8,5,4,4,0,2,0,1,1,\underbrace{0}_{6\ times})$	R_{14}^*
15			0.8	$(10,12,3,\underbrace{0}_{12\ times})$	R_{15}^*
16		20	0.09	$(2,4,0,2,0,2,0,2,1,0,1,1,0,1,1,2,0,0,1,0)$	R_{16}^*
17			0.4	$(4,8,3,2,2,0,1,\underbrace{0}_{13\ times})$	R_{17}^*
18			0.8	$(15,5,\underbrace{0}_{18\ times})$	R_{18}^*

6 Conclusion

In this paper, the estimation problem of the parameters is considered based on joint Type-II progressive censoring under competing risks model with random binomial removals when their life times follow GO distribution with different shape and scale parameters. The maximum likelihood estimators and their asymptotic confidence intervals are obtained. Furthermore, the Bayesian estimators and their corresponding credible intervals are also obtained by applying important sampling procedure. The performance of all the derived estimators are evaluated and compared via Monte carlo simulations. From the numerical simulation study, it is evident that the Bayes estimates based on importance sampling perform well when compared to the frequentist methods in terms of bias and mean square error. In addition, the credible intervals of the BEs are shorter than the classical intervals.

Table 3: Biases and MSEs of MLEs and BEs α_1 under Prior 0 and Prior 1 (between brackets).

C. S.		MLE	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R_1^*	Bias	0.5630	-0.4901(0.2553)	-0.4900(0.2107)	-0.4902(0.2199)	-0.4916(0.2746)	-0.4938(0.3246)
	MSE	0.8111	0.2402(0.0982)	0.2401(0.0703)	0.2403(0.0739)	0.2417(0.1322)	0.2439(0.1326)
R_2^*	Bias	0.5471	-0.4893(0.2324)	-0.4889(0.1892)	-0.4891(0.1996)	-0.4908(0.2132)	-0.4931(0.3058)
	MSE	0.7835	0.2394(0.0626)	0.2393(0.0292)	0.2395(0.0661)	0.2409(0.0887)	0.2432(0.1211)
R_3^*	Bias	0.4734	-0.4892(0.2077)	-0.4889(0.1396)	-0.4890(0.1515)	-0.4906(0.1678)	-0.4930(0.2555)
	MSE	0.6608	0.2391(0.0465)	0.2391(0.0237)	0.2392(0.0495)	0.2407(0.0673)	0.2431(0.0934)
R_4^*	Bias	0.4970	-0.3806(0.2454)	-0.3796(0.1435)	-0.3816(0.2188)	-0.3867(0.2461)	-0.3969(0.2747)
	MSE	0.4238	0.1489(0.1062)	0.1482(0.0568)	0.1495(0.0910)	0.1533(0.1116)	0.1609(0.1344)
R_5^*	Bias	0.4286	-0.3776(0.1831)	-0.3765(0.1345)	-0.3786(0.1554)	-0.3836(0.1612)	-0.3938(0.2136)
	MSE	0.3543	0.1472(0.0968)	0.1465(0.0500)	0.1478(0.0785)	0.1515(0.0862)	0.1590(0.1180)
R_6^*	Bias	0.3782	-0.3642(0.1628)	-0.3629(0.1025)	-0.3654(0.1359)	-0.3707(0.1423)	-0.3817(0.1919)
	MSE	0.3087	0.1379(0.0793)	0.1372(0.0439)	0.1387(0.0642)	0.1402(0.0650)	0.1501(0.0971)
R_7^*	Bias	0.4283	-0.1269(0.0457)	-0.1208(-0.0397)	-0.1329(-0.0461)	-0.1416(-0.0513)	-0.1747(-0.0724)
	MSE	0.2881	0.0592(0.0556)	0.0551(0.0536)	0.0563(0.0560)	0.0566(0.0570)	0.0628(0.0621)
R_8^*	Bias	0.3489	-0.1189(-0.0360)	-0.1130(0.0235)	-0.1247(0.0281)	-0.1330(0.0364)	-0.1642(-0.0549)
	MSE	0.2196	0.0478(0.0469)	0.0467(0.0460)	0.0470(0.0464)	0.0489(0.0480)	0.0536(0.0513)
R_9^*	Bias	0.3207	-0.0746(0.0258)	-0.0682(0.0027)	-0.0810(-0.0125)	-0.0888(-0.0175)	-0.1205(0.0201)
	MSE	0.1959	0.0467(0.0449)	0.0437(0.0430)	0.0444(0.0440)	0.04452(0.046)	0.0477(0.0461)

Continued.

C. S.		MLE	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R^*_{10}	Bias	0.4934	-0.3784(0.2192)	-0.3794(0.1351)	-0.3804(0.2105)	-0.3854(0.2301)	-0.3956(0.2739)
	MSE	0.4199	0.1483(0.0924)	0.1476(0.0532)	0.1489(0.0862)	0.1525(0.0878)	0.1600(0.1284)
R^*_{11}	Bias	0.3953	-0.3695(0.2030)	-0.3682(0.1227)	-0.3707(0.1658)	-0.3759(0.1763)	-0.3868(0.2315)
	MSE	0.3213	0.1415(0.0767)	0.1407(0.0445)	0.1422(0.0690)	0.1459(0.0791)	0.1539(0.1070)
R^*_{12}	Bias	0.3631	-0.3606(0.1192)	-0.3592(0.0872)	-0.3619(0.1156)	-0.3671(0.1389)	-0.3783(0.1713)
	MSE	0.2901	0.1358(0.0699)	0.1350(0.0420)	0.1366(0.0613)	0.1424(0.0619)	0.1479(0.0957)
R^*_{13}	Bias	0.3568	-0.1140(-0.0337)	-0.1083(-0.0175)	-0.1196(-0.0247)	-0.1277(-0.0312)	-0.1586(-0.0565)
	MSE	0.2257	0.0555(0.0526)	0.0541(0.0485)	0.0548(0.0504)	0.0564(0.0510)	0.0619(0.0554)
R^*_{14}	Bias	0.3352	-0.0930(-0.0326)	-0.0869(0.0169)	-0.0989(-0.0219)	-0.1062(-0.0287)	-0.1359(-0.0397)
	MSE	0.2098	0.0454(0.0430)	0.0438(0.0421)	0.0441(0.0433)	0.0447(0.0441)	0.0473(0.0462)
R^*_{15}	Bias	0.3172	-0.0331(-0.0052)	-0.0266(-0.0011)	-0.0397(0.0036)	-0.0467(0.0086)	-0.0778(-0.0143)
	MSE	0.1948	0.0447(0.0389)	0.0435(0.0375)	0.0436(0.0379)	0.0443(0.0381)	0.0449(0.0404)
R^*_{16}	Bias	0.3688	-0.0738(0.0222)	-0.0727(0.0111)	-0.0750(0.0392)	-0.0780(0.0503)	-0.0876(0.0697)
	MSE	0.2066	0.0415(0.0092)	0.0356(0.0090)	0.0446(0.0091)	0.0449(0.0094)	0.0777(0.0096)
R^*_{17}	Bias	0.3055	-0.0301(0.0085)	-0.0293(0.0073)	-0.0310(0.0080)	-0.0339(0.0082)	-0.0427(0.0091)
	MSE	0.1606	0.0275(0.0217)	0.0214(0.0206)	0.0265(0.0220)	0.0273(0.0236)	0.0283(0.0281)
R^*_{18}	Bias	0.2947	-0.0217(0.0037)	-0.0208(0.0036)	-0.0226(0.0044)	-0.0254(0.0077)	-0.0341(0.0086)
	MSE	0.1534	0.0181(0.0076)	0.0178(0.0057)	0.0184(0.0083)	0.0199(0.0087)	0.0242(0.0103)

Table 4: ALs and CPs of approximate and Bayes 95% CIs of α_1 when under Prior 0 and Prior 1(between brackets).

C. S.		Approx	SE	LINEX		EE	
				$\nu=-0.5$	$\nu=0.5$	$q=-0.5$	$q=0.5$
R_1^*	AL	2.2384	2.1941(0.8609)	2.1119(0.8140)	2.1206(0.7910)	2.1701(0.8303)	2.2899(0.9239)
	CP	0.7883	0.9420(0.9430)	0.9440(0.9450)	0.9430(0.9440)	0.9410(0.9420)	0.9400(0.9410)
R_2^*	AL	2.2056	2.0863(0.8483)	1.9985(0.7351)	2.0023(0.7808)	2.0587(0.8157)	2.1792(0.8994)
	CP	0.7935	0.9450(0.9460)	0.9470(0.9480)	0.9460(0.9470)	0.9430(0.9450)	0.9420(0.9440)
R_3^*	AL	2.0525	1.9948(0.8456)	1.9240(0.7271)	1.9265(0.7795)	1.9713(0.7985)	2.0792(0.8981)
	CP	0.7971	0.9460(0.9470)	0.9480(0.9490)	0.9470(0.9480)	0.9450(0.9460)	0.9430(0.9440)
R_4^*	AL	1.4238	0.9772(0.6977)	0.9324(0.6501)	0.9554(0.6873)	0.9616(0.6929)	1.0019(0.7175)
	CP	0.8439	0.9460(0.9470)	0.9480(0.9490)	0.9470(0.9480)	0.9450(0.9460)	0.9440(0.9450)
R_5^*	AL	1.3275	0.8943(0.6836)	0.9170(0.6231)	0.9259(0.6803)	0.8583(0.6883)	0.9266(0.7096)
	CP	0.8631	0.9470(0.9480)	0.9490(0.9510)	0.9480(0.9490)	0.9460(0.9470)	0.9450(0.9460)
R_6^*	AL	1.2570	0.8556(0.6499)	0.8077(0.6079)	0.8753(0.6354)	0.8758(0.6701)	0.8812(0.6900)
	CP	0.9139	0.9490(0.9500)	0.9510(0.9520)	0.9500(0.9510)	0.9480(0.9490)	0.9470(0.9480)
R_7^*	AL	1.0750	0.5997(0.2951)	0.5706(0.2671)	0.5755(0.2868)	0.6103(0.2899)	0.6551(0.2995)
	CP	0.9255	0.9500(0.9510)	0.9520(0.9530)	0.9510(0.9520)	0.9500(0.9510)	0.9480(0.9490)
R_8^*	AL	0.9854	0.5883(0.2855)	0.5603(0.2505)	0.5674(0.2839)	0.5990(0.2893)	0.6498(0.2911)
	CP	0.9335	0.9520(0.9530)	0.9540(0.9550)	0.9530(0.9540)	0.9510(0.9520)	0.9500(0.9510)
R_9^*	AL	0.9529	0.5898(0.2740)	0.5552(0.2395)	0.5668(0.2700)	0.5859(0.2709)	0.6351(0.2788)
	CP	0.9350	0.9540(0.9580)	0.9510(0.9520)	0.9560(0.9570)	0.9570(0.9590)	0.9530(0.9570)

Continued.

C. S.		Approx.	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R^*_{10}	AL	1.4189	0.9214(0.8574)	0.8824(0.7353)	0.9009(0.7931)	0.9079(0.8137)	0.9442(0.9196)
	CP	0.8411	0.9430(0.9440)	0.9470(0.9590)	0.9420(0.9450)	0.9390(0.9420)	0.9380(0.9410)
R^*_{11}	AL	1.2807	0.8832(0.8311)	0.8393(0.7222)	0.8638(0.7746)	0.8690(0.7983)	0.9050(0.8765)
	CP	0.8533	0.9440(0.9480)	0.9460(0.9580)	0.9400(0.9490)	0.9430(0.9480)	0.9450(0.9460)
R^*_{12}	AL	1.2350	0.8613(0.7623)	0.8184(0.6927)	0.8416(0.7161)	0.8465(0.7336)	0.8822(0.8166)
	CP	0.8641	0.9450(0.9500)	0.9480(0.9540)	0.9460(0.9510)	0.9480(0.9490)	0.9470(0.9480)
R^*_{13}	AL	0.9939	0.6967(0.3040)	0.6631(0.2752)	0.6439(0.2944)	0.6698(0.2994)	0.7096(0.3091)
	CP	0.9129	0.9480(0.9510)	0.9500(0.9530)	0.9490(0.9520)	0.9470(0.9500)	0.9460(0.9490)
R^*_{14}	AL	0.9602	0.6855(0.2795)	0.5975(0.2586)	0.6601(0.2717)	0.6684(0.2762)	0.7089(0.2830)
	CP	0.9320	0.9500(0.9520)	0.9520(0.9530)	0.9480(0.9510)	0.9490(0.9510)	0.9480(0.9500)
R^*_{15}	AL	0.9483	0.6422(0.2583)	0.5804(0.2356)	0.6207(0.2500)	0.6255(0.2552)	0.6623(0.2617)
	CP	0.9278	0.9500(0.9510)	0.9530(0.9540)	0.9490(0.9520)	0.9480(0.9510)	0.9470(0.9490)
R^*_{16}	AL	0.8709	0.4316(0.0311)	0.3887(0.0279)	0.4171(0.0226)	0.4280(0.0309)	0.4365(0.0312)
	CP	0.9393	0.9510(0.9530)	0.9520(0.9550)	0.9520(0.9540)	0.9510(0.9520)	0.9490(0.9500)
R^*_{17}	AL	0.8085	0.3883(0.0300)	0.3582(0.0207)	0.3803(0.0264)	0.3856(0.0298)	0.3920(0.0301)
	CP	0.9348	0.9540(0.9550)	0.9560(0.9570)	0.9530(0.9550)	0.9460(0.9520)	0.9490(0.9510)
R^*_{18}	AL	0.7983	0.3816(0.0263)	0.3514(0.0176)	0.3745(0.0225)	0.3808(0.0262)	0.3823(0.0264)
	CP	0.9390	0.9590(0.9610)	0.9600(0.9620)	0.9590(0.9600)	0.9580(0.9590)	0.9410(0.9590)

Table 5: Biases and MSEs of MLEs and BEs of α_2 under Prior 0 and Prior 1 (between brackets).

C. S.		MLE	SE	LINEX		EE	
				$\nu=-0.5$	$\nu=0.5$	$q=-0.5$	$q=0.5$
R_1^*	Bias	1.5544	-0.5975(-0.1067)	-0.5974(-0.0892)	-0.5976(-0.1249)	-0.5981(-0.1397)	-0.5990(-0.2054)
	MSE	3.3876	0.3570(0.0881)	0.3569(0.0877)	0.3571(0.0886)	0.3577(0.0932)	0.3588(0.1052)
R_2^*	Bias	1.2102	-0.5973(-0.0996)	-0.5972(-0.0863)	-0.5974(-0.1092)	-0.5980(-0.1198)	-0.5989(-0.1865)
	MSE	3.1978	0.3568(0.0774)	0.3567(0.0763)	0.3569(0.0764)	0.3576(0.0828)	0.3587(0.1012)
R_3^*	Bias	1.1425	-0.5970(-0.0697)	-0.5970(-0.0615)	-0.5970(-0.0772)	-0.5977(0.0969)	-0.5988(-0.1614)
	MSE	3.1447	0.3564(0.0687)	0.3564(0.0683)	0.3564(0.0688)	0.3573(0.0743)	0.3585(0.0912)
R_4^*	Bias	0.7437	-0.5833(-0.0645)	-0.5832(-0.0504)	-0.5834(-0.0677)	-0.5857(-0.0881)	-0.5895(-0.1604)
	MSE	1.2022	0.3406(0.0731)	0.3405(0.0726)	0.3407(0.0748)	0.3434(0.0783)	0.3478(0.0944)
R_5^*	Bias	0.7227	-0.5825(-0.0589)	-0.5824(-0.0448)	-0.5826(-0.0667)	-0.5851(-0.0878)	-0.5892(-0.1448)
	MSE	1.1538	0.3399(0.0644)	0.3398(0.0640)	0.3400(-0.0645)	0.3428(0.0684)	0.3475(0.0826)
R_6^*	Bias	0.6651	-0.5809(0.0479)	-0.5808(-0.0330)	-0.5810(-0.0614)	-0.5837(-0.0819)	-0.5881(-0.1350)
	MSE	1.0573	0.3380(0.0561)	0.3378(0.0514)	0.3381(0.0550)	0.3411(0.0633)	0.3462(0.0667)
R_7^*	Bias	0.6635	-0.5445(-0.0381)	-0.5440(0.0294)	-0.5450(-0.0561)	-0.5494(0.0740)	-0.5578(-0.1147)
	MSE	0.8629	0.3004(0.0383)	0.2999(0.0367)	0.3008(0.0396)	0.3054(0.0435)	0.3141(0.0495)
R_8^*	Bias	0.6486	-0.5392(-0.0373)	-0.5386(-0.0200)	-0.5398(0.0434)	-0.5446(-0.0716)	-0.5545(-0.0978)
	MSE	0.8336	0.2951(0.0328)	0.2946(0.0243)	0.2956(0.0276)	0.3006(0.0411)	0.3106(0.0480)
R_9^*	Bias	0.6320	-0.5391(0.0360)	-0.5384(-0.0138)	-0.5397(-0.0385)	-0.5443(-0.0578)	-0.5534(-0.0781)
	MSE	0.8023	0.2950(0.0252)	0.2944(0.0196)	0.2955(0.0229)	0.3003(0.0235)	0.3096(0.0472)

Continued.

C. S.		MLE	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R^*_{10}	Bias	0.7399	-0.5829(-0.0599)	-0.5828(-0.0311)	-0.5830(0.0659)	-0.5854(0.0822)	-05894(-0.1599)
	MSE	1.1930	0.3402(0.0639)	0.3400(0.0629)	0.3403(0.0662)	0.3430(0.0674)	0.3476(0.0807)
R^*_{11}	Bias	0.6926	-0.5817(-0.0480)	-0.5816(-0.0297)	-0.5818(-0.0595)	-0.5843(-0.0795)	-0.5884(-0.1230)
	MSE	1.0791	0.3389(0.0440)	0.3387(0.0392)	0.3390(0.0418)	0.3418(0.0562)	0.3465(0.0638)
R^*_{12}	Bias	0.6541	-0.5803(-0.0391)	-0.5802(-0.0295)	-0.5805(-0.0576)	-0.5830(0.0793)	-0.5874(-0.1130)
	MSE	1.0215	0.3375(0.0413)	0.3373(0.0369)	0.3376(0.0416)	0.3405(0.0493)	0.3454(0.0579)
R^*_{13}	Bias	0.6569	-0.5389(-0.0347)	-0.5383(-0.0293)	-0.5394(-0.0560)	-0.5441(-0.0791)	-0.5534(-0.0980)
	MSE	0.8497	0.2944(0.0525)	0.2938(0.0434)	0.2949(0.0466)	0.2997(0.0612)	0.3094(0.0631)
R^*_{14}	Bias	0.6389	-0.5387(0.0338)	-0.5381(0.0152)	-0.5393(-0.0459)	-0.5440(-0.0770)	-0.5532(-0.0796)
	MSE	0.8113	0.2940(0.0439)	0.2935(0.0375)	0.2946(0.0418)	0.2954(0.0503)	0.3087(0.0514)
R^*_{15}	Bias	0.5840	-0.5337(0.0316)	-0.5329(-0.0150)	-0.5344(-0.0294)	-0.5394(-0.0456)	-0.5494(-0.0494)
	MSE	0.7297	0.2898(0.0409)	0.2891(0.0353)	0.2904(0.0387)	0.2936(0.0478)	0.3054(0.0496)
R^*_{16}	Bias	0.6270	-0.4540(0.0289)	-0.4516(-0.0111)	-0.4562(-0.0278)	-0.4630(-0.0441)	-0.4796(-0.0489)
	MSE	0.7040	0.2256(0.0297)	0.2244(0.0214)	0.2267(0.0343)	0.2323(0.0385)	0.2450(0.0481)
R^*_{17}	Bias	0.6183	-0.4517(-0.0120)	-0.4496(-0.0092)	-0.4537(-0.0167)	-0.4604(-0.0348)	-0.4763(-0.0354)
	MSE	0.6844	0.2226(0.0168)	0.2214(0.0154)	0.2238(0.0238)	0.2295(0.0249)	0.2423(0.0391)
R^*_{18}	Bias	0.5747	-0.4461(0.0050)	-0.4435(0.0029)	-0.4484(0.0036)	-0.4557(0.0171)	-0.4729(-0.0041)
	MSE	0.6179	0.2190(0.0111)	0.2176(0.0103)	0.2203(0.0182)	0.2263(-0.0038)	0.2397(0.0302)

Table 6: ALs and CPs of approximate and Bayes 95% CI of α_2 under Prior 0 and Prior 1(between brackets).

C. S.		Approx	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R_1^*	AL	5.4871	0.8453(0.5679)	0.7418(0.5023)	0.8071(0.5453)	0.8206(0.5551)	0.9048(0.5826)
	CP	0.9000	0.9380(0.9450)	0.9400(0.9460)	0.9380(0.9440)	0.9370(0.9430)	0.9360(0.9420)
R_2^*	AL	5.3377	0.8391(0.5599)	0.7383(0.4888)	0.7953(0.5370)	0.8065(0.5460)	0.8810(0.5758)
	CP	0.9010	0.9430(0.9460)	0.9430(0.9470)	0.9410(0.9450)	0.9400(0.9440)	0.9390(0.9430)
R_3^*	AL	5.2991	0.8187(0.5524)	0.7135(0.4984)	0.7743(0.5354)	0.7864(0.5425)	0.8608(0.5633)
	CP	0.9133	0.9450(0.9470)	0.9460(0.9480)	0.9440(0.9460)	0.9430(0.9450)	0.9420(0.9440)
R_4^*	AL	3.1976	0.8181(0.2807)	0.6841(0.2348)	0.7685(0.2656)	0.7740(0.2761)	0.8600(0.2857)
	CP	0.9156	0.9480(0.9490)	0.9490(0.9500)	0.9470(0.9480)	0.9460(0.9470)	0.9450(0.9460)
R_5^*	AL	3.1508	0.7726(0.2574)	0.6761(0.2223)	0.7260(0.2457)	0.7433(0.2541)	0.8402(0.2609)
	CP	0.9188	0.9500(0.9510)	0.9510(0.9530)	0.9480(0.9520)	0.9480(0.9490)	0.9470(0.9480)
R_6^*	AL	3.0285	0.7624(0.2462)	0.6625(0.2106)	0.7257(0.2429)	0.7268(0.2339)	0.8244(0.2495)
	CP	0.9202	0.9520(0.9530)	0.9530(0.9550)	0.9510(0.9530)	0.9500(0.9530)	0.9500(0.9510)
R_7^*	AL	2.5240	0.7548(0.1034)	0.6573(0.0752)	0.7014(0.0935)	0.7196(0.1022)	0.7869(0.1046)
	CP	0.9251	0.9520(0.9540)	0.9540(0.9560)	0.9520(0.9540)	0.9510(0.9540)	0.9510(0.9530)
R_8^*	AL	2.4947	0.7207(0.0967)	0.6034(0.0711)	0.7001(0.0959)	0.6892(0.0887)	0.7379(0.0975)
	CP	0.9258	0.9530(0.9550)	0.9560(0.9570)	0.9540(0.9550)	0.9520(0.9540)	0.9510(0.9530)
R_9^*	AL	2.4546	0.6511(0.0791)	0.5844(0.0590)	0.6340(0.0719)	0.6343(0.0783)	0.6718(0.0799)
	CP	0.9277	0.9560(0.9570)	0.9570(0.9580)	0.9550(0.9560)	0.9530(0.9550)	0.9520(0.9540)

Continued.

C. S.		Approx	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R^*_{10}	AL	3.1904	0.8440(0.2601)	0.7129(0.2159)	0.8069(0.2562)	0.8089(0.2456)	0.8923(0.2640)
	CP	0.9102	0.9460(0.9480)	0.9470(0.9470)	0.9450(0.9460)	0.9440(0.9450)	0.9430(0.9440)
R^*_{11}	AL	3.0875	0.8377(0.2584)	0.6889(0.2253)	0.7824(0.2555)	0.7985(0.2471)	0.8765(0.2614)
	CP	0.9104	0.9470(0.9490)	0.9480(0.9480)	0.9460(0.9470)	0.9450(0.9460)	0.9440(0.9450)
R^*_{12}	AL	3.0092	0.7772(0.2499)	0.6250(0.2190)	0.7182(0.2471)	0.7306(0.2395)	0.8257(0.2529)
	CP	0.9105	0.9480(0.9490)	0.9490(0.9500)	0.9480(0.9480)	0.9470(0.9470)	0.9460(0.9470)
R^*_{13}	AL	2.5086	0.7496(0.0912)	0.6732(0.0704)	0.6968(0.0904)	0.7229(0.0840)	0.7970(0.0921)
	CP	0.9212	0.9490(0.9510)	0.9500(0.9540)	0.9490(0.9500)	0.9470(0.9510)	0.9460(0.9480)
R^*_{14}	AL	2.4752	0.7436(0.0875)	0.6557(0.0660)	0.6938(0.0866)	0.7111(0.0791)	0.8299(0.0884)
	CP	0.9254	0.9500(0.9520)	0.9520(0.9550)	0.9500(0.9510)	0.9490(0.9520)	0.9480(0.9490)
R^*_{15}	AL	2.3821	0.7428(0.0761)	0.6347(0.0586)	0.6899(0.0755)	0.7047(0.0697)	0.7988(0.0766)
	CP	0.9279	0.9520(0.9530)	0.9540(0.9560)	0.9520(0.9530)	0.9510(0.9520)	0.9490(0.9500)
R^*_{16}	AL	2.1509	0.5870(0.0151)	0.5462(0.0086)	0.5619(0.0124)	0.5724(0.0152)	0.6360(0.0153)
	CP	0.9395	0.9540(0.9550)	0.9550(0.9570)	0.9530(0.9540)	0.9520(0.9520)	0.9500(0.9510)
R^*_{17}	AL	2.1311	0.5261(0.0147)	0.4966(0.0076)	0.5149(0.0119)	0.5240(0.0146)	0.5660(0.0148)
	CP	0.9338	0.9550(0.9560)	0.9570(0.9580)	0.9540(0.9550)	0.9530(0.9550)	0.9520(0.9540)
R^*_{18}	AL	2.0639	0.4963(0.0113)	0.4742(0.0063)	0.4892(0.0094)	0.5050(0.0112)	0.5475(0.0113)
	CP	0.9306	0.9570(0.9580)	0.9580(0.9590)	0.9550(0.9560)	0.9540(0.9560)	0.9530(0.9550)

Table 7: Biases and MSEs of MLEs and BEs of β_1 under Prior 0 and Prior 1 (between brackets).

C. S.		MLE	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R_1^*	Bias	1.2303	-0.9991(0.8960)	-0.9991(0.8809)	-0.9991(0.8857)	-0.9992(0.8922)	-0.9994(-0.8987)
	MSE	1.2127	0.9983(0.8867)	0.9983(0.8608)	0.9983(0.8635)	0.9985(0.8831)	0.9988(0.8898)
R_2^*	Bias	1.1297	-0.9991(0.8767)	-0.9991(0.8515)	-0.9991(0.8798)	-0.9992(0.8893)	-0.9994(-0.8931)
	MSE	1.2113	0.9983(0.8620)	0.9983(0.8584)	0.9983(0.8478)	0.9985(0.8642)	0.9988(0.8738)
R_3^*	Bias	1.0972	-0.9991(0.8604)	-0.9991(0.8284)	-0.99911(-0.8768)	-0.9992(-0.8850)	-0.9993(0.8821)
	MSE	1.0949	0.9982(0.8490)	0.9983(0.8434)	0.9983(0.8443)	0.9985(0.8608)	0.9987(0.8675)
R_4^*	Bias	1.0865	-0.9975(0.8474)	-0.9975(-0.8275)	-0.9975(0.8621)	-0.9975(-0.8687)	-0.9976(-0.8767)
	MSE	1.0808	0.9949(0.8365)	0.9949(0.8348)	0.9949(0.8439)	0.9950(0.8528)	0.9953(0.8618)
R_5^*	Bias	1.0862	-0.9974(0.8408)	-0.9974(0.8247)	-0.9974(0.8520)	-0.9975(0.8578)	-0.9976(0.8604)
	MSE	1.0804	0.9949(0.8275)	-0.9949(0.8163)	0.9949(0.8310)	0.9950(0.8361)	0.9953(0.8515)
R_6^*	Bias	1.0754	-0.9974(0.8212)	0.9974(-0.8184)	-0.9974(0.8221)	-0.9975(0.8348)	-0.9976(-0.8362)
	MSE	1.0758	0.9949(0.8249)	0.9948(0.8108)	0.9949(0.8258)	0.9950(0.8340)	0.9953(0.8424)
R_7^*	Bias	1.0096	-0.9954(0.8083)	-0.9954(-0.7680)	-0.9955(0.7940)	-0.9955(0.8209)	-0.9956(0.8260)
	MSE	1.0603	0.9909(0.8060)	0.9909(0.7873)	0.9909(0.8247)	0.9910(0.8254)	0.9911(0.8317)
R_8^*	Bias	1.0007	-0.9954(-0.7651)	-0.9954(-0.7577)	-0.9954(0.8033)	-0.9954(-0.8165)	-0.9955(-0.8214)
	MSE	1.0513	0.9907(0.8008)	0.9907(0.7766)	0.9907(0.7965)	0.9909(0.8082)	0.9910(0.8235)
R_9^*	Bias	0.9977	-0.9954(0.7646)	-0.9954(-0.7042)	-0.9954(0.7927)	-0.9954(0.8009)	-0.9955(0.8123)
	MSE	1.0407	0.9907(0.7945)	0.9907(0.7564)	0.9907(0.7705)	0.9908(0.7843)	0.9910(0.8043)

Continued.

C. S.		MLE	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R^*_{10}	Bias	1.1294	-0.9975(0.6925)	-0.9975(0.6628)	-0.9975(0.6963)	-0.9975(0.7273)	-0.9976(0.7601)
	MSE	1.0786	0.9949(0.7852)	0.9949(0.7563)	0.9949(0.7739)	0.9951(0.8018)	0.9953(0.8504)
R^*_{11}	Bias	1.1252	-0.9974(0.6780)	-0.9974(0.6485)	-0.9974(0.6840)	-0.9975(0.7146)	-0.9976(0.7472)
	MSE	1.0768	0.9949(0.7817)	0.9949(0.7552)	0.9949(0.7636)	0.9950(0.7996)	0.9953(0.8389)
R^*_{12}	Bias	1.0871	-0.9974(0.6635)	-0.9974(0.6426)	-0.9974(0.6712)	-0.997470(0.6717)	-0.997591(0.6899)
	MSE	1.0728	0.9948(0.7772)	0.9948(0.7477)	0.9948(0.7563)	0.9949(0.7613)	0.9952(0.8269)
R^*_{13}	Bias	1.0769	-0.9954(0.6562)	-0.9954(0.6380)	-0.9954(0.6573)	-0.9954(0.6586)	-0.9955(0.6646)
	MSE	1.0687	0.9908(0.7611)	0.9908(0.7472)	0.9908(0.7509)	0.9909(0.7633)	0.9912(0.8021)
R^*_{14}	Bias	1.0755	-0.9953(0.6505)	-0.9953(0.6202)	-0.9953(0.6395)	-0.9954(0.6527)	-0.9955(0.6619)
	MSE	1.0635	0.9907(0.7547)	0.9907(0.7307)	0.9907(0.7348)	0.9908(0.7581)	0.9910(0.7975)
R^*_{15}	Bias	1.0708	-0.9954(0.6441)	-0.9954(0.6189)	-0.9953(0.6344)	-0.9954(0.6448)	-0.9955(0.6504)
	MSE	1.0481	0.9907(0.7370)	0.9907(0.7188)	0.9907(0.7249)	0.9908(0.7443)	0.9910(0.7850)
R^*_{16}	Bias	0.9965	-0.9931(0.6306)	-0.9931(0.5997)	-0.9931(0.6078)	-0.9931(0.6320)	-0.9932(0.6479)
	MSE	1.0430	0.9862(0.5044)	0.9862(0.4676)	0.9862(0.4871)	0.9863(0.5263)	0.9865(0.5797)
R^*_{17}	Bias	0.9958	-0.9930(0.6154)	-0.9930(0.5803)	-0.9930(0.5904)	-0.9931(0.6181)	-0.9932(0.6329)
	MSE	1.0332	0.9861(0.4912)	0.9861(0.4544)	0.9861(0.4739)	0.9862(0.5120)	0.9863(0.5601)
R^*_{18}	Bias	0.9949	-0.9930(0.6041)	-0.9930(0.5628)	-0.9930(0.5897)	-0.9930(0.6081)	-0.9931(0.6083)
	MSE	1.0187	0.9860(0.4516)	0.9860(0.4190)	0.9860(0.4367)	0.9861(0.4703)	0.9862(0.5142)

Table 8: ALs and CPs of approximate and Bayes 95% CIs of β_1 under Prior 0 and Prior 1(between brackets).

C. S.		Approx	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R_1^*	AL	2.0971	1.6429(0.0061)	1.6092(0.0060)	1.6549(0.0061)	1.6565(0.0061)	1.6586(0.0066)
	CP	0.5201	0.9390(0.9460)	0.9400(0.9470)	0.9410(0.9450)	0.9400(0.9430)	0.9350(0.9390)
R_2^*	AL	2.0328	1.6177(0.0060)	1.6082(0.0059)	1.6148(0.0060)	1.6192(0.0060)	1.6679(0.0060)
	CP	0.7312	0.9460(0.9460)	0.9470(0.9480)	0.9440(0.9460)	0.9430(0.9450)	0.9400(0.9420)
R_3^*	AL	1.6868	1.5854(0.0059)	1.5936(0.0059)	1.5778(0.0059)	1.5965(0.0059)	1.6461(0.0059)
	CP	0.7494	0.9470(0.9480)	0.9490(0.9490)	0.9470(0.9480)	0.9460(0.9470)	0.9410(0.9440)
R_4^*	AL	1.2813	1.2694(0.00493)	1.2553(0.0049)	1.2629(0.0049)	1.2645(0.0049)	1.3098(0.0049)
	CP	0.7772	0.9500(0.9510)	0.9510(0.9520)	0.9490(0.9500)	0.9470(0.9480)	0.9430(0.9450)
R_5^*	AL	0.9830	0.9253(0.0047)	0.9386(0.0046)	0.9182(0.0047)	0.9308(0.0047)	1.2782(0.0047)
	CP	0.9113	0.9510(0.9520)	0.9520(0.9530)	0.9510(0.9510)	0.9490(0.9500)	0.9440(0.9480)
R_6^*	AL	0.9713	0.9516(0.0033)	0.8274(0.0032)	0.8649(0.0032)	0.9106(0.0033)	1.0504(0.0033)
	CP	0.9185	0.9520(0.9530)	0.9540(0.9540)	0.9520(0.9530)	0.9510(0.9520)	0.9450(0.9500)
R_7^*	AL	0.9504	0.8263(0.0033)	0.7789(0.0031)	0.7959(0.0033)	0.8169(0.0032)	0.8673(0.0033)
	CP	0.9293	0.9520(0.9540)	0.9550(0.9560)	0.9530(0.9540)	0.9520(0.9530)	0.9510(0.9520)
R_8^*	AL	0.9073	0.8238(0.0046)	0.8136(0.0046)	0.8232(0.0046)	0.8251(0.0046)	(1.26510.0046)
	CP	0.9356	0.9520(0.9530)	0.9530(0.9560)	0.9530(0.9550)	0.9520(0.9540)	0.9520(0.9530)
R_9^*	AL	0.8672	0.7710(0.0032)	0.8042(0.0031)	0.7543(0.0031)	0.7751(0.0032)	0.7889(0.0032)
	CP	0.9377	0.9550(0.9560)	0.9560(0.9570)	0.9540(0.9560)	0.9530(0.9550)	0.9530(0.9540)

Continued.

C. S.		Approx	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R^*_{10}	AL	1.8897	1.3257(0.0048)	1.2910(0.0048)	1.2957(0.0048)	1.3249(0.0048)	1.3716(0.0048)
	CP	0.6193	0.9450(0.9470)	0.9460(0.9480)	0.9440(0.9450)	0.9430(0.9440)	0.9400(0.9430)
R^*_{11}	AL	1.6813	1.3106(0.0047)	1.2750(0.0047)	1.2767(0.0047)	1.2911(0.0047)	1.3462(0.0047)
	CP	0.7785	0.9470(0.9490)	0.9480(0.9490)	0.9470(0.9480)	0.9460(0.9470)	0.9450(0.9460)
R^*_{12}	AL	1.2801	1.2421(0.0046)	1.2092(0.0045)	1.2154(0.0045)	1.2255(0.0046)	1.2791(0.0046)
	CP	0.8483	0.9480(0.9510)	0.9490(0.9520)	0.9480(0.9490)	0.9480(0.9490)	0.9460(0.9480)
R^*_{13}	AL	1.1114	0.9828(0.0034)	0.8764(0.0033)	0.8993(0.0033)	0.9374(0.0034)	1.0816(0.0034)
	CP	0.8925	0.9510(0.9520)	0.9520(0.9540)	0.9500(0.9510)	0.9480(0.9500)	0.9470(0.9480)
R^*_{14}	AL	0.9820	0.8989(0.0032)	0.8365(0.0031)	0.8406(0.0032)	0.8682(0.0032)	0.9813(0.0032)
	CP	0.9230	0.9520(0.9530)	0.9530(0.9550)	0.9510(0.9520)	0.9500(0.9510)	0.9490(0.9510)
R^*_{15}	AL	0.9090	0.8022(0.0032)	0.8284(0.00312)	0.7987(0.0031)	0.8233(0.0031)	0.8026(0.0032)
	CP	0.9296	0.9520(0.9540)	0.9540(0.9560)	0.9530(0.9530)	0.9510(0.9520)	0.9500(0.9510)
R^*_{16}	AL	0.7992	0.5612(0.0016)	0.5312(0.0013)	0.5183(0.0015)	0.5441(0.0016)	0.6196(0.0016)
	CP	0.9354	0.9520(0.9550)	0.9550(0.9560)	0.9540(0.9550)	0.9530(0.9540)	0.9510(0.9520)
R^*_{17}	AL	0.7311	0.5218(0.0015)	0.5187(0.0013)	0.4930(0.0014)	0.5241(0.0015)	0.5659(0.0015)
	CP	0.9369	0.9550(0.9570)	0.9560(0.9580)	0.9540(0.9550)	0.9530(0.9550)	0.9520(0.9540)
R^*_{18}	AL	0.5635	0.5182(0.0014)	0.5177(0.0012)	0.4855(0.0013)	0.5122(0.0014)	0.5591(0.0014)
	CP	0.9393	0.9570(0.9590)	0.9580(0.9600)	0.9570(0.9590)	0.9560(0.9580)	0.9550(0.9560)

Table 9: Biases and MSEs of MLEs and BEs of β_2 under Prior 0 and Prior 1 (between brackets).

C. S.		MLE	SE	LINEX		EE	
				$\nu=-0.5$	$\nu=0.5$	$q=-0.5$	$q=0.5$
R_1^*	Bias	0.3831	-0.2996(0.2992)	-0.2996(0.2849)	-0.2996(0.2917)	-0.2997(0.2946)	-0.2998(0.2991)
	MSE	0.1263	0.0898(0.0794)	0.0898(0.0679)	0.0898(0.0823)	0.0898(0.0859)	0.0899(0.0936)
R_2^*	Bias	0.3657	-0.2996(0.2938)	-0.2996(0.2739)	-0.2996(0.2809)	-0.2996(0.2916)	-0.2998(0.2957)
	MSE	0.1189	0.0898(0.0715)	0.0898(0.0663)	0.0898(0.0680)	0.0898(0.0843)	0.0899(0.0902)
R_3^*	Bias	0.3630	-0.2996(0.2687)	-0.2996(0.2644)	-0.2996(0.2653)	-0.2997(0.2783)	-0.2999(-0.2824)
	MSE	0.1112	0.0898(0.0639)	0.0898(0.0630)	0.0898(0.0640)	0.0898(0.0658)	0.0899(0.0863)
R_4^*	Bias	0.3281	-0.2991(0.2633)	-0.2991(0.2549)	-0.2991(0.2583)	-0.2992(0.2650)	-0.2994(0.2800)
	MSE	0.1036	0.0895(0.0558)	0.0895(0.0688)	0.0895(0.0532)	0.0895(0.0569)	0.0896(0.0768)
R_5^*	Bias	0.3216	-0.2991(0.2542)	-0.2991(0.2416)	-0.2991(0.2482)	-0.2992(0.2564)	-0.2994(0.2691)
	MSE	0.1019	0.0895(0.0523)	0.0895(0.0407)	0.0895(0.0463)	0.0895(0.0542)	0.0896(0.0681)
R_6^*	Bias	0.3163	-0.2991(0.2419)	-0.2991(0.2413)	-0.2991(0.2389)	-0.2992(0.2426)	-0.2994(0.2627)
	MSE	0.0997	0.0895(0.0356)	0.0895(0.0241)	0.0895(0.0363)	0.0895(0.0419)	0.0896(0.0635)
R_7^*	Bias	0.3049	-0.2985(0.2374)	-0.2985(0.2348)	-0.2985(0.2389)	-0.2986(0.2412)	-0.2988(-0.2599)
	MSE	0.0981	0.0891(0.0249)	0.0891(0.0212)	0.0891(0.0289)	0.0892(0.0412)	0.0893(0.0421)
R_8^*	Bias	0.3027	-0.2985(0.2318)	-0.2985(0.2157)	-0.2985(0.2264)	-0.2986(0.2357)	-0.2988(0.2388)
	MSE	0.0932	0.0891(0.0235)	0.0891(0.0120)	0.0891(0.0203)	0.0892(0.0276)	0.0893(0.0326)
R_9^*	Bias	0.3003	-0.2985(0.2122)	-0.2985(0.2071)	-0.2985(-0.2121)	-0.2986(0.2182)	-0.2988(0.2261)
	MSE	0.0925	0.0891(0.0125)	0.0891(0.0105)	0.0891(0.0138)	0.0891(0.0157)	0.0893(0.0180)

Continued.

C. S.		MLE	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R^*_{10}	Bias	0.3488	-0.2991(0.2839)	-0.2991(0.2756)	-0.2991(0.2818)	-0.2992(0.2840)	-0.2994(0.2978)
	MSE	0.1215	0.0895(0.0732)	0.0895(0.0656)	0.0895(0.0723)	0.0895(0.0762)	0.0897(0.0810)
R^*_{11}	Bias	0.3394	-0.2991(0.2721)	-0.2991(0.2712)	-0.2991(0.2747)	-0.2992(0.2800)	-0.2994(0.2924)
	MSE	0.1156	0.0895(0.0621)	0.0895(0.0245)	0.0895(0.0588)	0.0895(0.0635)	0.0897(0.0664)
R^*_{12}	Bias	0.3361	-0.2991(0.2638)	-0.2991(0.2614)	-0.2991(0.2624)	-0.2992(0.2734)	-0.2994(0.2770)
	MSE	0.1039	0.0895(0.0413)	0.0895(0.0218)	0.0895(0.0470)	0.0895(0.0482)	0.0897(0.0529)
R^*_{13}	Bias	0.3340	-0.2985(0.2515)	-0.2985(0.2444)	-0.2985(0.2514)	-0.2986(0.2571)	-0.2988(0.2729)
	MSE	0.1006	0.0891(0.0307)	0.0891(0.0188)	0.0891(0.0351)	0.0891(0.0410)	0.0893(0.0439)
R^*_{14}	Bias	0.3231	-0.2985(0.2360)	-0.2985(0.2355)	-0.2985(0.2393)	-0.2986(0.2424)	-0.2988(0.2514)
	MSE	0.1004	0.0891(0.0250)	0.0891(0.0166)	0.0891(0.0293)	0.0892(0.0316)	0.0893(0.0409)
R^*_{15}	Bias	0.3197	-0.2985(0.2302)	-0.2985(0.2128)	-0.2985(0.2306)	-0.2986(0.2374)	-0.2987(0.2450)
	MSE	0.0981	0.0891(0.0156)	0.0891(0.0120)	0.0891(0.0135)	0.0891(0.0138)	0.0892(0.0181)
R^*_{16}	Bias	0.3162	-0.2978(0.2186)	-0.2978(0.2100)	-0.29780(0.2280)	-0.2979(0.2283)	-0.2981(0.2395)
	MSE	0.0958	0.0887(0.0056)	0.0887(0.0052)	0.0887(0.0062)	0.0888(0.0075)	0.0889(0.0123)
R^*_{17}	Bias	0.3120	-0.2978(0.2084)	-0.2978(0.2025)	-0.29780(0.2123)	-0.2979(0.2211)	-0.2981(0.2236)
	MSE	0.0921	0.0887(0.0052)	0.0887(0.0048)	0.0887(0.0054)	0.0887(0.0057)	0.0889(0.0097)
R^*_{18}	Bias	0.3087	-0.2978(0.2010)	-0.2978(0.2008)	-0.2978(0.2010)	-0.2979(0.2070)	-0.2980(-0.2107)
	MSE	0.0902	0.0887(0.0041)	0.0887(0.0040)	0.0887(0.0043)	0.0887(0.0048)	0.0888(0.0089)

Table 10: ALs and CPs of approximate and Bayes 95% CIs of β_2 under Prior 0 and Prior 1(between brackets).

C. S.		Approx	SE	LINEX		EE	
				$\nu = -0.5$	$\nu = 0.5$	$q = -0.5$	$q = 0.5$
R_1^*	AL	1.7876	1.6019(0.0051)	1.4841(0.0048)	1.5408(0.0051)	1.5563(0.0051)	1.6579(0.0051)
	CP	0.7639	0.9430(0.9450)	0.9450(0.9460)	0.9440(0.9450)	0.9410(0.9440)	0.9390(0.9430)
R_2^*	AL	1.5935	1.4477(0.0050)	1.3645(0.004757)	1.3845(0.004959)	1.4131(0.0049)	1.5041(0.0050)
	CP	0.8234	0.9440(0.9460)	0.9460(0.9470)	0.9450(0.9460)	0.9430(0.9450)	0.9400(0.9440)
R_3^*	AL	1.2548	1.1438(0.0047)	1.0849(0.0046)	1.0908(0.0047)	1.1098(0.004706)	1.1822(0.004739)
	CP	0.8754	0.9440(0.9470)	0.9470(0.9480)	0.9450(0.9470)	0.9440(0.9460)	0.9420(0.9450)
R_4^*	AL	1.2135	1.1429(0.0040)	1.0322(0.0036)	1.0947(0.0039)	1.1115(0.0040)	1.1618(0.0040)
	CP	0.9022	0.9450(0.9480)	0.9480(0.9490)	0.9460(0.9480)	0.9450(0.9470)	0.9430(0.9470)
R_5^*	AL	1.0814	1.0019(0.0038)	0.9048(0.0035)	0.9553(0.0037)	0.9631(0.0038)	1.0673(0.0038)
	CP	0.9055	0.9460(0.9480)	0.9480(0.9490)	0.9480(0.9480)	0.9470(0.9480)	0.9450(0.9470)
R_6^*	AL	0.9390	0.8666(0.0036)	0.7887(0.0034)	0.8307(0.0036)	0.8345(0.0036)	0.8984(0.0037)
	CP	0.9058	0.9470(0.9490)	0.9490(0.9500)	0.9490(0.9490)	0.9480(0.9500)	0.9460(0.9480)
R_7^*	AL	0.7890	0.4458(0.0024)	0.4124(0.0020)	0.4286(0.0024)	0.4323(0.0024)	0.4705(0.0024)
	CP	0.9124	0.9480(0.9500)	0.9500(0.9510)	0.9490(0.9500)	0.9490(0.9500)	0.9470(0.9490)
R_8^*	AL	0.7599	0.4169(0.0024)	0.4046(0.0020)	0.4091(0.0023)	0.4127(0.0023)	0.4172(0.0024)
	CP	0.9229	0.9500(0.9510)	0.9520(0.9520)	0.9500(0.9510)	0.9500(0.9510)	0.9490(0.9500)
R_9^*	AL	0.7374	0.3968(0.0023)	0.3842(0.0020)	0.3856(0.0023)	0.3921(0.0022)	0.4314(0.0023)
	CP	0.9321	0.9520(0.9550)	0.9530(0.9540)	0.9520(0.9530)	0.9520(0.9530)	0.9500(0.9510)

Continued.

C. S.		Approx	SE	LINEX		EE	
				$\nu=-0.5$	$\nu=0.5$	$q=-0.5$	$q=0.5$
R^*_{10}	AL	1.1134	1.0471(0.0037)	0.9478(0.0035)	1.0139(0.0036)	1.0305(0.0037)	1.0961(0.0037)
	CP	0.8830	0.9440(0.9450)	0.9470(0.9470)	0.9430(0.9450)	0.9420(0.9440)	0.9400(0.9410)
R^*_{11}	AL	1.0090	0.9169(0.0037)	0.8186(0.0034)	0.8786(0.0037)	0.8801(0.0036)	0.9575(0.0037)
	CP	0.8881	0.9460(0.9480)	0.9480(0.9490)	0.9440(0.9490)	0.9440(0.9480)	0.9410(0.9460)
R^*_{12}	AL	0.9537	0.6809(0.0036)	0.6364(0.0034)	0.6624(0.0036)	0.6716(0.0037)	0.6988(0.0037)
	CP	0.8944	0.9500(0.9510)	0.9490(0.9530)	0.9460(0.9500)	0.9450(0.9490)	0.9430(0.9480)
R^*_{13}	AL	0.8496	0.5016(0.0024)	0.4389(0.0020)	0.4712(0.0023)	0.4756(0.0024)	0.5284(0.0024)
	CP	0.9054	0.9510(0.9540)	0.9500(0.9550)	0.9470(0.9520)	0.9460(0.9500)	0.9440(0.9510)
R^*_{14}	AL	0.7748	0.3925(0.0024)	0.3807(0.0020)	0.3846(0.0023)	0.3883(0.0024)	0.4013(0.0024)
	CP	0.9129	0.9520(0.9550)	0.9500(0.9560)	0.9490(0.9540)	0.9480(0.9530)	0.9470(0.9510)
R^*_{15}	AL	0.7440	0.3773(0.0023)	0.3642(0.0020)	0.3672(0.0022)	0.3717(0.0023)	0.3884(0.0023)
	CP	0.9229	0.9520(0.9560)	0.9510(0.9570)	0.9500(0.9550)	0.9490(0.9540)	0.9480(0.9530)
R^*_{16}	AL	0.6911	0.2514(0.0012)	0.2395(0.0006)	0.2459(0.0011)	0.2461(0.0012)	0.2531(0.0012)
	CP	0.9275	0.9530(0.9570)	0.9540(0.9580)	0.9530(0.9570)	0.9520(0.9550)	0.9510(0.9540)
R^*_{17}	AL	0.6384	0.2310(0.0012)	0.2184(0.0008)	0.2245(0.0010)	0.2258(0.0012)	0.2343(0.0012)
	CP	0.9297	0.9540(0.9580)	0.9550(0.9590)	0.9540(0.9580)	0.9530(0.9570)	0.9520(0.9560)
R^*_{18}	AL	0.6256	0.2102(0.0010)	0.2158(0.0008)	0.2040(0.0008)	0.2101(0.0010)	0.2150(0.0010)
	CP	0.9351	0.9550(0.9590)	0.9560(0.9600)	0.9550(0.9590)	0.9540(0.9580)	0.9530(0.9570)

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