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Inferences for generalized Topp-Leone distribution under order statistics with application to polyester fibers data

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In this note, we obtain recurrence relations for the moments of order statistics from generalized Topp-Leone distribution without any restriction for the shape parameter. Several relations are also, derived when the shape parameter is an integer. In addition, we use these moments to obtain the best linear unbiased estimates of the location and scale parameters based on Type-II right-censored samples. In addition, we carry out some numerical illustrations through Monte Carlo simulations to show the usefulness of the findings. Finally, we apply the findings of the paper to some real data set.

keywords: Generalized Topp-Leone distribution, moments, order statistics, best linear unbiased estimators.

1 Introduction

In the last six decades or so, we see a spur in the efforts of order statistics, which are applied successfully to almost every possible sphere of human activity. Also, the moments of order statistics play a vital role in a wide range of theoretical and practical problems such the derivation of best linear unbiased estimators for scale and location-scale families of distributions based on complete and type-II censored samples, characterization of probability distributions, goodness-of fit tests, entropy estimation, analysis of censored samples, quality control, reliability etc. For example, it is seen that when the duration of the failed items is high, the reliability of an item is also high, which in turn makes the product too costly, both in terms of time and money. In such a situation the one may not

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know enough about the item in a short period of time and hence would require few early failures data for predicting the failure of future items. Thus moments of order statistics are useful in making these kinds of prediction in such situations. The early applications order statistics were concerned with empirical economic studies and coordination among various projects and efficient utilization of future emergencies.

Generalized Topp-Leone distribution is a univariate continuous two parameter distribution with bounded support which was first proposed and used as a model for tissue damage proportions data by Shekhawat and Sharma (2021). The hazard function of this model allows increasing and bathtub shaped hazard function such as Weibull and exponentiated exponential distributions; and so on and also show that GTL model is a better model for fitting tissue damage proportions data over the Beta, Kumaraswamy, unit-Weibull and unit-gamma distributions. They also study the properties and estimate the unknown parameters by using maximum likelihood method and they also, developed a parametric regression model to study the significance of the concentration level of drug/chemical on the tissue damage proportions in blood of the distribution.

Several researchers have worked in the field of order statistics have appeared in the literature, see Kamps (1991), Balakrishnan and Cohan (1991), Mohie El-Din et al. (1991), Sultan and Balakrishnan (000a), Sultan and Balakrishnan (000b), Mahmoud et al. (2005), Çetinkaya and Genç (2018), Genc (2012), MirMostafaee (2014), Kumar (2015), Balakrishnan et al. (2015), Sultan and AL-Thubyani (2016), Kumar and Dey (2017), Kumar and Goyal (019a), Kumar and Goyal (019b), Kumar et al. (2022), and many others. Kumar et al. (2017), Kumar et al. (2020) established exact explicit expression and recurrence relations for single and product moments of order statistics from the extended exponential and Type-II exponentiated log-logistic distribution. They also obtained the best linear unbiased estimator of the scale parameter based on complete and type-II right censored samples. Ahsanullah and Alzaatreh (2018) have obtained the moments of order statistics and also, estimate the unknown parameters of the Log-logistic distribution based on order statistics.

Recently, Shekhawat and Sharma (2021) proposed a generalization of the Topp-Leone distribution called generalized Topp-Leone (GTL) distribution with probability density function (pdf)

$$f(x;\delta,\xi) \quad = \quad 2\delta\xi x^{\delta\xi-1}\left(1-x^{\delta}\right)\left(2-x^{\delta}\right)^{\xi-1},\ 0<x<1,\ \delta,\ \xi>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) and quantile function are, respectively given by

$$F(x;\delta,\xi)=\left(x^{\delta}\left(2-x^{\delta}\right)\right)^{\xi},\ 0<x<1,\ \delta,\ \xi>0. \tag{2}$$

and let $F(x_p;\delta,\xi)=p$, $p\in\{0,1\}$, then quantile function is

$$x_p=\left\{1-\sqrt{1-p^{\frac{1}{\xi}}}\right\}^{\frac{1}{\delta}},$$

The kth moment of the GTL distribution in (1), can be easily computed as

$$\mu_k^{'} = \xi 2^{2\xi+\frac{k}{\delta}} \left\{ B_{1/2}\left(\xi + \frac{k}{\delta},\ \xi\right) - 2B_{1/2}\left(\xi + 1 + \frac{k}{\delta},\ \xi\right) \right\}, \tag{3}$$

where $B_x(.,.)$ denotes the incomplete beta function and defined by $B_x(\delta, \xi) = \int_0^x t^{\delta-1}(1-t)^{\xi-1}dt$

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics from the GTL distribution given in Equation (1) with its cdf in Equation (2). Then, the pdf of the uth order statistic is

$$f_{X_{u:n}}(x) = \frac{n!}{(u-1)!(n-u)!} F^{u-1}(x) \left[1 - F(x)\right]^{n-u} f(x),\ 0 < x < 1, \tag{4}$$

and the joint pdf of the uth and vth order statistics is

$$\begin{aligned} f_{X_{u:n},X_{v:n}}(x,y) &= \frac{n!}{(u-1)!(v-u-1)!(n-v)!} F^{u-1}(x) \left[F(y) - F(x)\right]^{v-1-u} \\ &\times \left[1 - F(y)\right]^{n-v} f(x)f(y),\ 0 < x < y,\ u,v = 1,2,\ldots,\ u < v. \end{aligned} \tag{5}$$

The purpose of this article is two fold. First we derive some exact and explicit expressions for single and product moments of order statistics of the GTL distribution. Next, we estimate the best linear unbised estimate of location and scale parameters of the model based on order statistics for different sample sizes and different parameter values and, also obtain the variances and covariances of the best linear unbised estimate for the GTL distribution, which we think would be of deep interest to applied statisticians.

In this paper, we derive the exact expressions for the single moments of order statistics from GTL distribution in Sections 2. An application of the theoretical results of Section 2 to the lifetimes of coherent systems is provided in Section 3. Section 4 is devoted to the product moments of order statistics from GTL distribution. In Section 5, we obtain BLUEs for the location and scale parameters by using these moments. A real data application is provided in Section 6. Finally, in Section 7 , we draw a conclusion for the paper.

2 Relations for single moments of order statistics

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics from the GTL distribution given in Equation (1) with its cdf in Equation (2). Then, the pdf of the uth order statistic is

$$\begin{aligned} f_{X_{u:n}}(x) &= \frac{2n!\delta\xi\, x^{\delta\xi-1}}{(u-1)!(n-u)!} \left[x^{\delta}(2-x^{\delta})\right]^{\xi(u-1)} \\ &\times \left[1 - \left\{x^{\delta}(2-x^{\delta})\right\}^{\xi}\right]^{n-u} (1-x^{\delta})(2-x^{\delta})^{\xi-1}. \end{aligned} \tag{6}$$

The kth moments of uth order statistics can be obtain from (6) as

$$\begin{aligned}
\mu_{u:n}^{(k)} &= \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{k+2\delta\xi(u+j)}{\delta}} \\
&\times \left\{B_{1/2}\left(\frac{k+\delta\xi(u+j)}{\delta}, \xi(u+j)\right) - 2B_{1/2}\left(\frac{k+\delta\xi(u+j)+\delta}{\delta}, \xi(u+j)\right)\right\} \quad (7)
\end{aligned}$$

Note that when $u = n = 1$, $\mu_{1:1}^{(k)} = \xi 2^{2\xi+\frac{k}{\delta}}\left\{B_{1/2}\left(\xi+\frac{k}{\delta},\ \xi\right) - 2B_{1/2}\left(\xi+1+\frac{k}{\delta},\ \xi\right)\right\}$, which agrees with (3). In addition from (7), the first and second moments of order statistics are, respectively, given by

$$\begin{aligned}
\mu_{u:n}^{(1)} &= \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{1+2\delta\xi(u+j)}{\delta}} \\
&\times \left\{B_{1/2}\left(\frac{1+\delta\xi(u+j)}{\delta}, \xi(u+j)\right) - 2B_{1/2}\left(\frac{1+\delta\xi(u+j)+\delta}{\delta}, \xi(u+j)\right)\right\} \quad (8)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{u:n}^{(2)} &= \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{2+2\delta\xi(u+j)}{\delta}} \\
&\times \left\{B_{1/2}\left(\frac{2+\delta\xi(u+j)}{\delta}, \xi(u+j)\right) - 2B_{1/2}\left(\frac{2+\delta\xi(u+j)+\delta}{\delta}, \xi(u+j)\right)\right\} \quad (9)
\end{aligned}$$

Theorem 1. For $\frac{r}{\delta} = m$,$m \in Z^+$, the rth moment of uth order statistics can be expressed as,

$$\begin{aligned}
\mu_{u:n}^{(r)} &= \frac{n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}\frac{(-1)^j}{(u+j)}\left[\frac{(-1)^{m+1}(-m-\xi(u+j)-1)_{m+1}}{(\xi(u+j))_{m+1}}\right. \\
&- 2^{m+2\xi(u+j)-1}(m+2)Be(\xi(u+j), m+\xi(u+j)+1) \\
&- \left.\sum_{k=1}^{m}(-1)^k(k-1)\frac{(-m-\xi(u+j))_k}{(\xi(u+j))_k}\right]. \quad (10)
\end{aligned}$$

Proof. For proving the above result, we will use the following relations,

(i) $\quad F(\delta,\xi;1+\xi;u) = \xi u^{-\xi} Be(\xi, 1-\delta; u),$

(ii) $\quad Be(m+v, v, \frac{1}{2}) = \frac{2^{-(m+v)}}{v} F(1-v, m+v; 1+m+v; \frac{1}{2}),$

(iii) $\quad F(\delta,\xi;y;u) = (1-u)^{-\delta} F(\delta, y-\xi; y; \frac{-u}{1-u}),$

(iv) $\quad F(\delta,\xi;y;u) = \sum_{k=0}^{\infty}\frac{(\delta)_k(\xi)_k u^k}{(\gamma)_k k!}, (a)_k = a(a+1)\ldots(a+k-1),$

(v) $\quad F(a,1,l-a,-1) = 2^{l-2a-2}\frac{\Gamma(1-a)\Gamma(l-a)}{\Gamma(l-2a)} - \frac{1}{2}\sum_{k=1}^{l-2}(-1)^k\frac{(1-l+a)_k}{(1-a)_k}.$

On using the relations (i)-(iii) in (7) and simplifying, we get

$$\begin{aligned} \mu_{u:n}^{(r)} &= \frac{n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}\frac{2(-1)^j}{u+j} \\ &\times \left[F(1-\xi(u+j),1,1+m+\xi(u+j),-1)(I_1)\right. \\ &- \left.F(1-\xi(u+j),1,2+m+\xi(u+j),-1)(I_2)\right] \end{aligned} \tag{11}$$

The terms I_1 and I_2 in (11) can be simplified using the relations (iv) and (v) as

$$\begin{aligned} I_1 &= 2^{m+2\xi(u+j)-2}\frac{\Gamma(\xi(u+j))\Gamma(1+m+\xi(u+j))}{\Gamma(m+2\xi(u+j))} \\ &- \frac{1}{2}\sum_{k=1}^{m}(-1)^k\frac{(-m-\xi(u+j))_k}{(\xi(u+j))_k} \\ I_2 &= 2^{m+2\xi(u+j)-1}\frac{\Gamma(\xi(u+j))\Gamma(2+m+\xi(u+j))}{\Gamma(1+m+2\xi(u+j))} \\ &- \frac{1}{2}\sum_{k=1}^{m+1}(-1)^k\frac{(-m-\xi(u+j)-1)_k}{(\xi(u+j))_k}. \end{aligned}$$

Substituting the above expressions of I_1 and I_2 in (11) we get (10).

We derive the following relation for single moments under the restriction that $0<\xi<1$ so that $m=1/\xi$ is a positive integer. If $m+1\le u\le n-1$ for all positive integer $n\ge 3$.

Theorem 2. Let $0<\xi<1$ so that $m=1/\xi$ is a positive integer. If $m+1\le u\le n-1$ for all positive integer $n\ge 3$, then we have the following moment relation for $k=1,2,\ldots$.

$$\begin{aligned} \mu_{u:n}^{(k-1)} &= C\Bigg\{\left(1-\frac{\delta-1}{k+\delta-1}\right)\left[\mu_{u-m+1:n-m}^{(k+\delta-1)}-\mu_{u-m:n-m}^{(k+\delta-1)}\right] \\ &+ \left(1-\frac{2\delta-1}{k+2\delta-1}\right)\left[\mu_{u-m:n-m}^{(k+2\delta-1)}-\mu_{u-m+1:n-m}^{(k+2\delta-1)}\right]\Bigg\}, \end{aligned}$$

where $k\in N$ and

$$C = \frac{2\delta n!(u-m)!}{km(u-1)!(n-m)!}.$$

Proof. We have

$$\mu_{u:n}^{(k-1)} = \frac{n!}{(u-1)!(n-u)!}\int_0^1 x^{k-1}F^{u-1}(x)\left[1-F(x)\right]^{n-u}f(x)dx.$$

Using integration by parts by treating x^{k-1} for integration and $F^{u-1}(x)\left[1-F(x)\right]^{n-u}f(x)$ for differentiation and noting that

$$f^2(x) = 2\delta\xi(1-x^\delta)x^{\delta-1}F^{1-m}(x)f(x)$$

and

$$f^{'}(x) = 2\delta(\xi-1)(1-x^{\delta})x^{\delta-1}F^{-m}(x)f(x) + 2\delta\xi\left[(\delta-1)x^{\delta-2}-(2\delta-1)x^{2\delta-2}\right]F^{1-m}(x)$$

we have

$$\begin{aligned}
\mu_{u:n}^{(k-1)} &= C_{u:n}\frac{x^k}{k}F^{u-1}(x)\left[1-F(x)\right]^{n-u}f(x)|_0^1 \\
&- C_{u:n}\int_0^1\frac{x^k}{k}\Big\{\Big[(u-1)F^{u-2}(x)\left[1-F(x)\right]^{n-u} \\
&- (n-u)F^{u-1}(x)\left[1-F(x)\right]^{n-u-1}\Big]f^2(x)+F^{u-1}(x)\left[1-F(x)\right]^{n-u}f^{'}(x)\Big\}dx \\
&= \frac{2\delta(1-\xi u)C_{u:n}}{k}\int_0^1 x^{k+\delta-1}(1-x^{\delta})F^{u-m-1}(x)\left[1-F(x)\right]^{n-u}f(x)dx \\
&+ \frac{2\delta\xi(n-u)C_{u:n}}{k}\int_0^1 x^{k+\delta-1}(1-x^{\delta})F^{u-m}(x)\left[1-F(x)\right]^{n-u-1}f(x)dx \\
&+ \frac{2\delta\xi C_{u:n}}{k}\int_0^1 x^{k+\delta-2}\Big[(2\delta-1)x^{\delta}-(\delta-1)\Big]F^{u-m}(x)\left[1-F(x)\right]^{n-u}dx. \qquad (12)
\end{aligned}$$

After some manipulation the first two integrals above will be

$$\frac{2\delta(1-\xi u)n!(u-m-1)!}{k(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(k+\delta-1)}-\mu_{u-m:n-m}^{(k+2\delta-1)}\right\}$$

and

$$\frac{2\delta\xi n!(u-m)!}{k(u-1)!(n-m)!}\left\{\mu_{u-m+1:n-m}^{(k+\delta-1)}-\mu_{u-m+1:n-m}^{(k+2\delta-1)}\right\},$$

respectively. The third integral in (12) will be

$$\frac{2\delta\xi(1-2\delta)n!(u-m)!}{k(k+2\delta-1)(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(k+2\delta-1)}-\mu_{u-m+1:n-m}^{(k+2\delta-1)}\right\} + \frac{2\delta\xi(\delta-1)n!(u-m)!}{k(k+\delta-1)(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(k+\delta-1)}-\mu_{u-m+1:n-m}^{(k+\delta-1)}\right\}.$$

after integrating by parts with some manipulation. Now substituting these results into (12) completes the proof. □

The recurrence relation without any restriction is presented by the following theorem.

Theorem 3. Let $X_1,\ldots,X_n$ be a random sample of size n from the GTL distribution, and let $X_{1:n}\leq X_{2:n}\leq\cdots\leq X_{n:n}$ be the corresponding order statistics, then for $1\leq u<n$ and $k\in N$, we have

$$\mu_{u+1:n}^{(k+\delta)} = \left(1+\frac{\delta}{k}\right)\mu_{u+1:n}^{(k)}+\left(\frac{k+\delta}{2\delta\xi u}+1\right)\mu_{u:n}^{(k+\delta)}-\left(\frac{k+\delta}{\delta\xi u}+\frac{\delta}{k}+1\right)\mu_{u:n}^{(k)}. \qquad (13)$$

Proof. We have

$$2\mu_{u:n}^{(k)} - \mu_{u:n}^{(k+\delta)} = \frac{n!}{(u-1)!(n-u)!}\int_0^1 x^{k-1}F^{u-1}(x)\left[1-F(x)\right]^{n-u}(2x - x^{\delta+1})f(x)dx$$
$$= \frac{2\delta\xi\, n!}{(u-1)!(n-u)!}\int_0^1 x^{k-1}(1-x^{\delta})F^{u}(x)\left[1-F(x)\right]^{n-u}dx,$$

where the last equality is obtained from (1) and (2) note that

$$(2x - x^{\delta+1})f(x) = 2\delta\xi(1-x^{\delta})F(x). \tag{14}$$

Upon integrating by parts by treating $x^{k-1}(1-x^{\delta})$ for integration and $F^{u}(x)\left[1-F(x)\right]^{n-u}$ for differentiation, we have

$$2\mu_{u:n}^{(k)} - \mu_{u:n}^{(k+\delta)} = 2\frac{\delta\xi\, n!}{(u-1)!(n-u)!}\Big\{\left(\frac{1}{k} - \frac{x^{\delta}}{k+\delta}\right)x^{k}F^{u}(x)\left[1-F(x)\right]^{n-u}\Big|_0^1$$
$$- \; u\int_0^1\left(\frac{1}{k} - \frac{x^{\delta}}{k+\delta}\right)x^{k}F^{u-1}(x)\left[1-F(x)\right]^{n-u}f(x)dx$$
$$+ \; (n-u)\int_0^1\left(\frac{1}{k} - \frac{x^{\delta}}{k+\delta}\right)x^{k}F^{u}(x)\left[1-F(x)\right]^{n-u-1}f(x)dx\Big\}. \tag{15}$$

The right hand side of (15) is zero if $u < n$. Therefore, it can be concluded that

$$\frac{1}{2\delta\xi u}\left(2\mu_{u:n}^{(k)} - \mu_{u:n}^{(k+\delta)}\right) = \int_0^1\left(\frac{1}{k} - \frac{x^{\delta}}{k+\delta}\right)x^{k}f_{X_{u+1:n}}(x;\delta,\xi)dx$$
$$- \int_0^1\left(\frac{1}{k} - \frac{x^{\delta}}{k+\delta}\right)x^{k}f_{X_{u:n}}(x;\delta,\xi)dx$$
$$= \frac{\mu_{u+1:n}^{(k)} - \mu_{u:n}^{(k)}}{k} - \frac{\mu_{u+1:n}^{(k+\delta)} - \mu_{u:n}^{(k+\delta)}}{k+\delta}$$

and the result follows from simplifying the above equation.

Remark 1. *Under the assumptions of Theorem 1 and for $k, n \in N$, we have*

$$\mu_{n:n}^{(k+\delta)} = \left(\frac{1}{k+\delta} + \frac{1}{2n\delta\xi}\right)^{-1}\left\{\left(\frac{1}{k} + \frac{1}{n\delta\xi}\right)\mu_{n:n}^{(k)} - \frac{\delta}{k(k+\delta)}\right\}. \tag{16}$$

The proof of the relation (16) is similar to that of Theorem 1.

Next, by using the techniques which is introduced by Thomas and Samuel (2008), if we assume that ξ is a positive integer value to obtain the several recurrence relations for single moments of lower sample sizes for GTL distribution.

Theorem 4. Suppose that $k, \xi \in N$, then we have the following results
(i) If $2 \le u \le n$, then

$$\mu_{u:n}^{(k)} = \frac{n}{u-1}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^{r}2^{\xi-r}\mu_{u-1:n-1}^{(k+\delta(\xi+r))}. \tag{17}$$

(ii) If $1 \leq u < n$, then

$$\mu_{u:n}^{(k)} = \frac{n}{n-u}\left\{\mu_{u:n-1}^{(k)} - \sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\mu_{u:n-1}^{(k+\delta(\xi+r))}\right\}. \quad (18)$$

Proof. We only prove part (ii) as the other part can be proven similarly. Since ξ is a positive integer value, the cdf of the GTL distribution can be represented as

$$F(x) = \sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}x^{\delta(\xi+r)}, \quad 0 < x < 1. \quad (19)$$

Therefore if $1 \leq u \leq n-1$, then we may write

$$\begin{aligned}\mu_{u:n}^{(k)} &= \frac{n!}{(u-1)!(n-u)!}\int_0^1 x^k\left\{1 - \sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}x^{\delta(\xi+r)}\right\}\\ &\times F^{u-1}(x)\left[1-F(x)\right]^{n-u-1}f(x)dx,\end{aligned}$$

then the result follows on simplifying the above expression. □

3 Application to the lifetime of coherent systems

The relations obtained in the preceding sections allow us to evaluate the expected lifetimes of coherent systems. The study of coherent systems is a very important area in reliability. Samaniego (1985) proposed the concept of signature of a coherent system that has become a very useful tool to compute the system reliability and expected lifetime. Consider a system with n components whose lifetimes, denoted as $X_1, X_2, \ldots, X_n$, are independent identically distributed (iid) random variables. Moreover, let Z be the lifetime of the whole system and $s_u = Pr(Z = X_{u:n})$ for $u = 1, \ldots, n$. Then for $p \in N$, we have (Samaniego (1985))

$$E(Z^p) = \sum_{u=1}^{n}\mathbf{s_u}\,\mu_{u:u}^{(p)}, \quad (20)$$

and the vector $\mathbf{s} = (s_1, \ldots, s_n)$ is called the signature vector.
Navarro et al. (2007) proved that for a coherent system with exchangeable components, we can write

$$E(Z^p) = \sum_{u=1}^{n}\lambda_u\,\mu_{1:u}^{(p)} = \sum_{u=1}^{n}\theta_u\,\mu_{u:u}^{(p)}, \quad (21)$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, are called minimal signature and maximal signature, respectively.

Now, suppose that a coherent system has n components such that the component lifetimes, $X_1, X_2, \ldots, X_n$, are iid random variables from generalized Topp-Leone distribution, then we can apply the relations obtained in section 3, to evaluate the expected system lifetime. For instance, from (16) and (21) and for $p \in \mathbb{N}$, we have

$$E(Z^{p+\delta}) = \sum_{u=1}^{n} \theta_u \left(\frac{1}{p+\delta} + \frac{1}{2u\delta\xi}\right)^{-1} \left\{ \left(\frac{1}{p} + \frac{1}{u\delta\xi}\right) \mu_{u:u}^{(p)} - \frac{\delta}{p(p+\delta)} \right\},$$

or from (18) and (20), for $\xi, p \in \mathbb{N}$, we can state

$$E(Z^{p}) = \sum_{u=1}^{n} \frac{ns_u}{n-u} \left\{ \mu_{u:n-1}^{(p)} - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u:n-1}^{(p+\delta(\xi+r))} \right\}.$$

4 Relations for product moments of order statistics

Under the condition $\xi \in \mathbb{N}$, we derive the closed expression for the simple product moments of two order statistics of GTL distribution as follows:

Theorem 5. For $1 \le u < v \le n$, $n \in N$ and $\xi \in N$ we have,

$$\begin{aligned} \mu_{u,v:n} &= C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \binom{n-v}{r} \binom{v-u-1}{s} (-1)^{r+s} \frac{1}{(s+u)\Delta_{r,s}} \mu_{s+u:s+u} \mu_{\Delta_{r,s}:\Delta_{r,s}} \\ &+ C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \sum_{t=0}^{\xi\Delta_{r,s}-1} \binom{n-v}{r} \binom{v-u-1}{s} \binom{\xi\Delta_{r,s}-1}{t} (-1)^{r+s+t} \frac{2^{\xi\Delta_{r,s}-t}\delta\xi}{(s+u)} \\ &\times \left\{ \frac{1}{1+\delta+\delta t+\delta\xi\Delta_{r,s}} \mu_{s+u:s+u}^{2+\delta+\delta t+\delta\xi\Delta_{r,s}} - \frac{1}{1+\delta t+\delta\xi\Delta_{r,s}} \mu_{s+u:s+u}^{2+\delta t+\delta\xi\Delta_{r,s}} \right\}, \end{aligned} \tag{22}$$

where $\Delta_{r,s} = v - u - s + r$.

Proof. Using the joint pdf of uth and vth order statistics, we have

$$\begin{aligned} \mu_{u,v:n} &= C_{u,v:n} \int_0^1 \int_x^1 xyF^{u-1}(x) \left[F(y) - F(x)\right]^{v-1-u} \left[1 - F(y)\right]^{n-v} f(x)f(y)dxdy \\ &= C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \binom{n-v}{r} \binom{v-u-1}{s} (-1)^{r+s} \\ &\times \int_0^1 xF^{s+u-1}(x)f(x)I(x)dx \end{aligned} \tag{23}$$

where

$$\begin{aligned} I(x) &= \int_x^1 y[F(y)]^{v-u-s+r-1} f(y)dy \\ &= \frac{1}{v-u-s+r}\mu_{v-u-s+r:v-u-s+r} - \int_0^x y[F(y)]^{v-u-s+r-1} f(y)dy \\ &= \frac{1}{\Delta_{r,s}}\mu_{\Delta_{r,s}:\Delta_{r,s}} + \delta\xi 2^{\xi\Delta_{r,s}}\left[\int_0^1 x^{\delta\xi\Delta_{r,s}+\delta+1}u^{\delta\xi\Delta_{r,s}+\delta}\left[1-\frac{1}{2}(xu)^\delta\right]^{\xi\Delta_{r,s}-1} du \right.\\ &- \left.\int_0^1 x^{\delta\xi\Delta_{r,s}+1}u^{\delta\xi\Delta_{r,s}}\left[1-\frac{1}{2}(xu)^\delta\right]^{\xi\Delta_{r,s}-1} du\right]. \qquad (24) \end{aligned}$$

Now substituting the values of $I(x)$ in after using some manipulations we obtain the desired result. □

Next, we obtained the explicit expression for the (k,l)th product moment without any restriction for ξ which includes the simple product moments as well. desired result.

Theorem 6. For $1 \le u < v \le n$ and $n, k, l \in \mathbb{N}$, we have

$$\begin{aligned} \mu_{u,v:n}^{(k,l)} &= \frac{\xi^2\, n!}{(u-1)!(v-u-1)!(n-v)!}\sum_{r=0}^{n-v}\sum_{s=0}^{v-u-1}\sum_{q=0}^{\infty}\binom{n-v}{r}\binom{v-u-1}{s}\binom{\xi(s+u)-1}{q} \\ &\times (-1)^{r+s+q}\, 2^{\frac{l+k+2\delta\xi(v+r)}{\delta}}\left\{\frac{1}{\eta_{u,k,\delta,\xi}(s,q)}B_{1/2}(\eta^*_{v,k,l,\delta,\xi}(r,q), \xi\Delta_{r,s})\right. \\ &- \frac{2(2\eta_{u,k,\delta,\xi}(s,q)+1)}{\eta_{u,k,\delta,\xi}(s,q)(\eta_{u,k,\delta,\xi}(s,q)+1)}B_{1/2}(\eta^*_{v,k,l,\delta,\xi}(r,q)+1, \xi\Delta_{r,s}) \\ &+ \left.\frac{4}{\eta_{u,k,\delta,\xi}(s,q)+1}B_{1/2}(\eta^*_{v,k,l,\delta,\xi}(r,q)+2, \xi\Delta_{r,s})\right\}. \qquad (25) \end{aligned}$$

where $\eta_{u,k,\delta,\xi}(s,q) = \frac{k}{\delta} + \xi(s+u) + q$ and $\eta^*_{v,k,l,\delta,\xi}(r,q) = \frac{l+k}{\delta} + \xi(v+r) + q$. If $\xi \in \mathbb{N}$, then the third summation stops at $q = \xi(s+u) - 1$.

Proof. Using (5), the (k,l)-th product moment of (u,v)th order statistics can be written as

$$\begin{aligned} \mu_{u,v:n}^{(k,l)} &= C_{u,v:n}\int_0^1\int_0^y x^k y^l F^{u-1}(x)\left[F(y)-F(x)\right]^{v-1-u}\left[1-F(y)\right]^{n-v} f(x)f(y)dxdy \\ &= C_{u,v:n}\sum_{r=0}^{n-v}\sum_{s=0}^{v-u-1}\binom{n-v}{r}\binom{v-u-1}{s}(-1)^{r+s} \\ &\times \int_0^1 y^l f(y)[F(y)]^{r,s-1} I(y)dy, \qquad (26) \end{aligned}$$

where

$$\begin{aligned} I(y) &= \int_0^y x^k [F(x)]^{s+u-1} f(x)dx \\ &= 2\delta\xi \int_0^y x^{k+\delta\xi(s+u)-1}(2-x^\delta)^{\xi(s+u)-1}(1-x^\delta)dx \\ &= \xi 2^{\frac{k}{\delta}+2\xi(s+u)} \int_0^{y^\delta/2} z^{\frac{k}{\delta}+\xi(s+u)-1}(1-z)^{\xi(s+u)-1}(1-2z)dz \\ &= \xi 2^{\frac{k}{\delta}+2\xi(s+u)} \sum_{q=0}^{\infty} \binom{\xi(s+u)-1}{q}(-1)^q \int_0^{y^\delta/2} z^{\eta_{u,k,\delta,\xi}(s,q)-1}(1-2z)dz \\ &= \xi \sum_{q=0}^{\infty} 2^{\xi(s+u)-q} \binom{\xi(s+u)-1}{q}(-1)^q \left(\frac{1}{\eta_{u,k,\delta,\xi}(s,q)} - \frac{y^\delta}{\eta_{u,k,\delta,\xi}(s,q)+1}\right) \\ &\times \; y^{\delta\eta_{u,k,\delta,\xi}(s,q)}. \end{aligned}$$

The above summation stops at $\xi(s+u)-1$ if $\xi \in \mathbb{N}$. Now substituting the values of $I(y)$ in (26) and after simplification, we get the result given in (25). □

Next, we present a recurrence relation for the product moments of order statistics for GTL distribution.

Theorem 7. For the GTL distribution $1 \le u \le v-2, v \le n$ and $k, l \in \mathbb{N}$, we have

$$\begin{aligned} \mu_{u+1,v:n}^{(k+\delta,l)} &= \left(1+\frac{\delta}{k}\right)\mu_{u+1,v:n}^{(k,l)} + \left(\frac{k+\delta}{2\delta\xi u}+1\right)\mu_{u,v:n}^{(k+\delta,l)} \\ &- \left(\frac{k+\delta}{\delta\xi u}+\frac{\delta}{k}+1\right)\mu_{u,v:n}^{(k,l)}. \end{aligned} \tag{27}$$

Proof. Form (5) and (14), we get

$$2\mu_{u,v:n}^{(k,l)} - \mu_{u,v:n}^{(k+\delta,l)} = \frac{n!}{(u-1)!(v-u-1)!(n-v)!}\int_0^1 y^l[1-F(y)]^{n-v} f(y)G(y)dy,$$

where

$$\begin{aligned} G(y) &= \int_0^y x^{k-1}[F(x)]^{u-1}[F(y)-F(x)]^{v-u-1} x(2-x^\delta) f(x)dx \\ &= 2\delta\xi \int_0^y x^{k-1}(1-x^\delta)[F(x)]^u [F(y)-F(x)]^{v-u-1}dx. \end{aligned}$$

Upon integrating by parts we have

$$\begin{aligned} G(y) &= 2\delta\xi \Bigg\{ \left(\frac{1}{k}-\frac{x^\delta}{k+\delta}\right) x^k F^u(x)\left[F(y)-F(x)\right]^{v-u-1}\Big|_0^y \\ &- u\int_0^y \left(\frac{1}{k}-\frac{x^\delta}{k+\delta}\right) x^k F^{u-1}(x)\left[F(y)-F(x)\right]^{v-u-1} f(x)dx \\ &+ (v-u-1)\int_0^y \left(\frac{1}{k}-\frac{x^\delta}{k+\delta}\right) x^k F^u(x)\left[F(y)-F(x)\right]^{v-u-2} f(x)dx \Bigg\}. \end{aligned} \tag{28}$$

If $v > u-1$, then the right hand side of (28) is equal to zero and as a consequence we have

$$\begin{aligned}\frac{1}{2\delta\xi u}\left(2\mu_{u,v:n}^{(k,l)}-\mu_{u,v:n}^{(k+\delta,l)}\right) &= \int_0^1\int_0^y\left(\frac{1}{k}-\frac{x^\delta}{k+\delta}\right)x^k y^l f_{X_{u+1,v:n}}(x,y;\delta,\xi)dxdy \\ &- \int_0^1\int_0^y\left(\frac{1}{k}-\frac{x^\delta}{k+\delta}\right)x^k y^l f_{X_{u,v:n}}(x,y;\delta,\xi)dxdy \\ &= \frac{\mu_{u+1,v:n}^{(k,l)}-\mu_{u,v:n}^{(k,l)}}{k}-\frac{\mu_{u+1,v:n}^{(k+\delta,l)}-\mu_{u,v:n}^{(k+\delta,l)}}{k+\delta},\end{aligned}$$

and hence the result follows. □

Remark 2. *For $1 \leq u \leq n-1$ and $k,l \in \mathbb{N}$, we have*

$$\mu_{u,u+1:n}^{(k+\delta,l)} = -\left(\frac{k+\delta}{2\delta\xi u}+1\right)^{-1}\left\{\left(1+\frac{\delta}{k}\right)\mu_{u+1:n}^{(k+l)}-\mu_{u+1:n}^{(k+\delta+l)}-\left(\frac{k+\delta}{\delta\xi u}+\frac{\delta}{k}+1\right)\mu_{u,u+1:n}^{(k,l)}\right\}.$$

Next, by using the techniques which is introduced by Thomas and Samuel (2008), if we assume that ξ is a positive integer value to prove the several recurrence relations for product moments of of lower sample sizes for GTL distribution.

Theorem 8. Let $\xi \in \mathbb{N}$, then we have
(i) If $2 \leq u < v \leq n$, then

$$\mu_{u,v:n}^{(k,l)} = \frac{n}{u-1}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\mu_{u-1,v-1:n-1}^{(k+\delta(\xi+r),l)}. \tag{29}$$

(ii) If $1 \leq u < v-1 \leq n-1$, then

$$\mu_{u,v:n}^{(k,l)} = \frac{n}{v-u-1}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r\, 2^{\xi-r}\left\{\mu_{u,v-1:n-1}^{(k,l+\delta(\xi+r))}-\mu_{u,v-1:n-1}^{(k+\delta(\xi+r),l)}\right\}. \tag{30}$$

(iii) If $1 \leq u < v \leq n-1$, then

$$\mu_{u,v:n}^{(k,l)} = \frac{n}{n-v}\left\{\mu_{u,v:n-1}^{(k,l)}-\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\mu_{u,v:n-1}^{(k,l+\delta(\xi+r))}\right\}. \tag{31}$$

Proof. We only prove part (ii). The other parts can be proven similarly. If $1 \leq u < v-1 \leq n-1$, then using the equation (13) and expanding the term $F(y)-F(x)$, we may write

$$\begin{aligned}
\mu_{u,v:n}^{(k,l)} &= C_{u,v:n}\int_0^1\int_0^y x^k y^l F^{u-1}(x)\left[F(y)-F(x)\right]^{v-1-u}\left[1-F(y)\right]^{n-v} f(x)f(y)dxdy \\
&= C_{u,v:n}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\int_0^1\int_0^y x^k y^{l+\delta(\xi+r)} \\
&\times\ F^{u-1}(x)\left[F(y)-F(x)\right]^{v-u-2}\left[1-F(y)\right]^{n-v} f(x)f(y)dxdy \\
&-\ C_{u,v:n}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\int_0^1\int_0^y x^{k+\delta(\xi+r)} y^{l} \\
&\times\ F^{u-1}(x)\left[F(y)-F(x)\right]^{v-u-2}\left[1-F(y)\right]^{n-v} f(x)f(y)dxdy.
\end{aligned}$$

Upon simplifying the above expression, we get the result. □

5 Estimation of the location and scale parameters

Here, we study parameter estimation for the GTL distribution based on order statistics.

5.1 BLUEs of of the location and scale parameters

Let $Z_1 \le Z_2 \le \cdots \le Z_m$ be a random sample of size m from GTL distribution with the pdf of scale-parameter GTL distribution is

$$f(z;\delta,\xi,b) = \frac{2\delta\xi}{b}\left(\frac{x}{b}\right)^{\delta\xi-1}\left[1-\left(\frac{x}{b}\right)^{\delta}\right]\left[2-\left(\frac{x}{b}\right)^{\delta}\right]^{\xi-1}, \quad (32)$$

where $0 < z < 1,\ \delta,\ \xi, b > 0$. The pdf of the location-scale parameter is

$$f(z;\delta,\xi,a,b) = \frac{2\delta\xi}{b}\left(\frac{x-a}{b}\right)^{\delta\xi-1}\left[1-\left(\frac{x-a}{b}\right)^{\delta}\right]\left[2-\left(\frac{x-a}{b}\right)^{\delta}\right]^{\xi-1}, \quad (33)$$

where $0 < z < 1,\ \delta,\ \xi, a, b > 0$. Let $Z_{1:m} \le Z_{2:m} \le \cdots \le Z_{n-c:m},\ \ c = 0, 1, \ldots, n-1$ denote Type-II right-censored sample from the location-scale parameter GTL distribution in Equation (33). Let us define $Y_{u:m} = (Z_{u:m} - a)/b,\ E(Y_{u:m}) = a_{u;m}^{(1)},\ 1 \le u \le (m-c)$, and $Cov(Y_{u:m},\ Y_{v:m}) = b_{u,v:m} = a_{u,v:m}^{(1,1)} - a_{u:m}^{(1)}\ a_{v:m}^{(1)},\ 1 \le u < v \le (m-c)$. Therefore

$$\mathbf{Z} = (Z_{1:m}, Z_{2:m}, \ldots, Z_{(m-c):m})^T,$$

$$\mathbf{a} = (a_{1:m}, a_{2:m}, \ldots, a_{(m-c):m})^T,$$

$$\mathbf{1} = \underbrace{(1,\ 1,\ \ldots,\ 1)^T}_{m-c},$$

and

$$\sum = ((b_{u,v})), \quad 1 \leq u, \quad v \leq m - c,$$

where $a_{u:m} = E(Z_{u:m})$, $b_{uu} = Var(Z_{u:m})$ and $b_{uv} = Cov(Y_{u:m}, Y_{v:m})$, $u = 1, 2, \ldots (m-c)$. Therefore the BLUEs of a and b of GTL distribution can be obtained as follows [see Arnold et al. (1992)]

$$a^* = \sum_{u=1}^{m-c} p_u Z_{u:m} \quad \text{and} \quad b^* = \sum_{u=1}^{m-c} q_u Z_{u:m},$$

where

$$p_u = \left\{ \frac{\mathbf{a^T} \sum^{-1} \mathbf{a1^T} \sum^{-1} - \mathbf{a^T} \sum^{-1} \mathbf{1a^T} \sum^{-1}}{(\mathbf{a^T} \sum^{-1} \mathbf{a})(\mathbf{1^T} \sum^{-1} \mathbf{1}) - (\mathbf{a^T} \sum^{-1} \mathbf{1})^2} \right\}, \tag{34}$$

$$q_u = \left\{ \frac{\mathbf{1^T} \sum^{-1} \mathbf{1a^T} \sum^{-1} - \mathbf{1^T} \sum^{-1} \mathbf{a1^T} \sum^{-1}}{(\mathbf{a^T} \sum^{-1} \mathbf{a})(\mathbf{1^T} \sum^{-1} \mathbf{1}) - (\mathbf{a^T} \sum^{-1} \mathbf{1})^2} \right\}, \tag{35}$$

and, the variances and covariance of these BLUEs of GTL distribution can be obtained as follows

$$Var(a^*) = b^2 \left\{ \frac{\mathbf{a^T} \sum^{-1} \mathbf{a}}{(\mathbf{a^T} \sum^{-1} \mathbf{a})(\mathbf{1^T} \sum^{-1} \mathbf{1}) - (\mathbf{a^T} \sum^{-1} \mathbf{1})^2} \right\} = b^2 \eta_1, \tag{36}$$

$$Var(b^*) = b^2 \left\{ \frac{\mathbf{1^T} \sum^{-1} \mathbf{1}}{(\mathbf{a^T} \sum^{-1} \mathbf{a})(\mathbf{1^T} \sum^{-1} \mathbf{1}) - (\mathbf{a^T} \sum^{-1} \mathbf{1})^2} \right\} = b^2 \eta_2, \tag{37}$$

and

$$Cov(a^*, \ b^*) = b^2 \left\{ \frac{-\mathbf{a^T} \sum^{-1} \mathbf{1}}{(\mathbf{a^T} \sum^{-1} \mathbf{a})(\mathbf{1^T} \sum^{-1} \mathbf{1}) - (\mathbf{a^T} \sum^{-1} \mathbf{1})^2} \right\} = b^2 \eta_3. \tag{38}$$

The values of p_u and q_u are presented in Tables 1 and 2 for different values of sample sizes $m = 7, \ 10$ and different censoring cases $c = 0(1)([n/2] - 1)$ and for some selected values for $\upsilon = 1, 2$. The coefficient of the BLUEs p_u and q_u given by (34) and (35) respectively, the conditions,

$$\sum_{u=1}^{n-c} p_u = 1$$

and

$$\sum_{u=1}^{n-c} q_u = 0,$$

which are used to check the computations accuracy.

6 Data analysis

To demonstrate how the proposed methods can be used in practice, we consider the following real-life data set (see Quesenberry and Hales (1980)). The data set related to 30 measurements of tensile strength of polyester fibers. The data are:

```
0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169,
0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463,
0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926
```

Now a random sample of size 10 is selected from the given data set and data are: 0.105, 0.432, 0.642, 0.529, 0.069, 0.361, 0.887, 0.674, 0.216, 0.081. By using the GTL in Eq. (1) for the given sample, we have the maximumlikelihood estimate of $\delta_{ML} = 2.9967$ and $\xi_{ML} = 0.32063$. Figure 1 shows Q–Q plot of the sample. The Kolmogorov-Smirnov (K-S) statistic is 0.15679 and the corresponding p-value is 0.9357. This shows the suitability of the GTL distribution for this data set.

Then by using the BLUEs coefficients in Tables 3 and 4, we have

$$a^* = \sum_{u=1}^{n} p_u X_{u:n} = 0.030836 \quad \text{and} \quad b^* = \sum_{u=1}^{n} q_u X_{u:n} = 1.329179$$

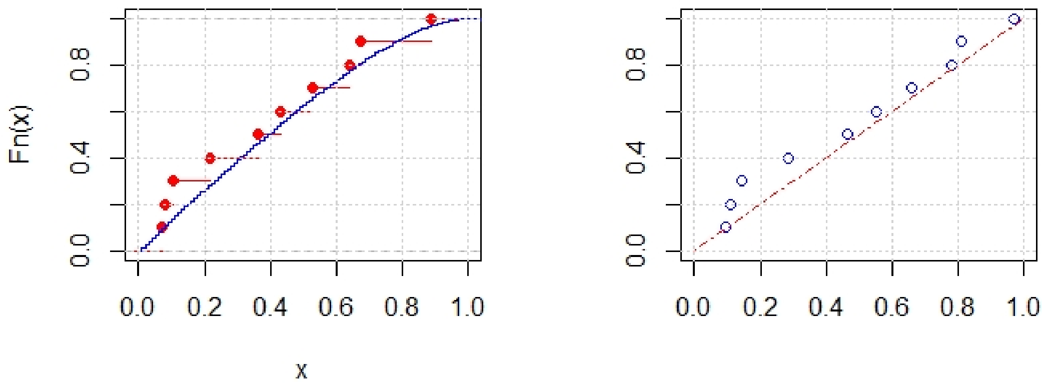

Figure 1: ECDF and Q-Q plot of the real sample based on GTL distribution

7 Conclusion

In this paper, the single and product moments of the order statistics from the GTL distribution are derived in explicit forms. The single and product moments are used to obtain the BLUEs of the location and scale parameters of GTL distribution. The variances and covariances are calculated to show the performance of the BLUEs. Finally, one real data set has been used to obtain the MLEs of the model parameters and BLUEs of a and b. From our finding, we see that the moments of order statistics of the distribution are well behaved. This will encourage the study of the other properties of order statistics for a future research.

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Table 1: First moments, second moments and variances of the ith order statistics from GTL distribution for $r = 1, 2, \ldots, 15$, $\delta = 0.5$ and $\xi = 1, 2$

δ	ξ	u	n	$E(X)$	$E(X^2)$	$V(X)$	δ	ξ	u	n	$E(X)$	$E(X^2)$	$V(X)$
0.5	1	1	1	0.166667	0.066667	0.038889	0.5	2	1	1	0.266667	0.119048	0.047936
			2	0.066667	0.014286	0.009841				2	0.146032	0.040115	0.018790
			3	0.035714	0.004762	0.003486				3	0.098968	0.019297	0.009503
			4	0.022222	0.002020	0.001526				4	0.074313	0.011134	0.005612
			5	0.015152	0.000999	0.000769				5	0.059259	0.007173	0.003661
			6	0.010989	0.000550	0.000429				6	0.049158	0.004975	0.002559
			7	0.008333	0.000327	0.000257				7	0.041931	0.003638	0.001880
			8	0.006536	0.000206	0.000164				8	0.036516	0.002768	0.001435
			9	0.005263	0.000137	0.000109				9	0.032312	0.002173	0.001128
			10	0.004329	0.000094	0.000075				10	0.028957	0.001748	0.000909
		2	2	0.266667	0.119048	0.047936			2	2	0.387302	0.197980	0.047977
			3	0.128571	0.033333	0.016803				3	0.240160	0.081752	0.024075
			4	0.076191	0.012987	0.007182				4	0.172933	0.043787	0.013882
			5	0.050505	0.006105	0.003554				5	0.134527	0.026979	0.008881
			6	0.035964	0.003247	0.001953				6	0.109765	0.018161	0.006113
			7	0.026923	0.001885	0.001161				7	0.092518	0.012996	0.004437
			8	0.020915	0.001170	0.000732				8	0.079841	0.009728	0.003353
			9	0.016718	0.000764	0.000484				9	0.070146	0.007535	0.002615
			10	0.013671	0.000520	0.000333				10	0.062500	0.005998	0.002091
		3	3	0.335714	0.161905	0.049201			3	3	0.460873	0.256094	0.043690
			4	0.180952	0.053680	0.020936				4	0.307387	0.119716	0.025229
			5	0.114719	0.023310	0.010150				5	0.230541	0.069000	0.015851
			6	0.079587	0.011822	0.005487				6	0.184053	0.044615	0.010740
			7	0.058566	0.006650	0.003220				7	0.152882	0.031072	0.007699
			8	0.044947	0.004033	0.002012				8	0.130547	0.022803	0.005760
			9	0.035604	0.002589	0.001322				9	0.113775	0.017401	0.004456
			10	0.028909	0.001739	0.000904				10	0.100731	0.013686	0.003540
		4	4	0.387302	0.197980	0.047977			4	4	0.512034	0.301553	0.039374
			5	0.225108	0.073926	0.023252				5	0.358617	0.153526	0.024920
			6	0.149850	0.034799	0.012343				6	0.277030	0.093385	0.016639
			7	0.107615	0.018717	0.007136				7	0.225614	0.062673	0.011771
			8	0.081265	0.011013	0.004409				8	0.190107	0.044854	0.008713
			9	0.063634	0.006919	0.002870				9	0.164091	0.033608	0.006682
			10	0.051224	0.004573	0.001949				10	0.144212	0.026068	0.005270
		5	5	0.427850	0.228993	0.045938			5	5	0.550389	0.338560	0.035632
			6	0.262737	0.093490	0.024459				6	0.399411	0.183597	0.024068
			7	0.181527	0.046860	0.013908				7	0.315592	0.116419	0.016820
			8	0.133964	0.026420	0.008474				8	0.261122	0.080492	0.012307
			9	0.103304	0.016129	0.005458				9	0.222626	0.058911	0.009349
			10	0.082249	0.010440	0.003675				10	0.193910	0.044918	0.007317

Table 1: Continued.

δ	ξ	u	n	$E(X)$	$E(X^2)$	$V(X)$	δ	ξ	u	n	$E(X)$	$E(X^2)$	$V(X)$
		6	6	0.460873	0.256094	0.043690			6	6	0.580584	0.369553	0.032474
			7	0.295221	0.112142	0.024986				7	0.432938	0.210469	0.023033
			8	0.210064	0.059124	0.014997				8	0.348274	0.137975	0.016680
			9	0.158492	0.034653	0.009533				9	0.291918	0.097756	0.012540
			10	0.124359	0.021819	0.006354				10	0.251343	0.072904	0.009731
		7	7	0.488481	0.280086	0.041472			7	7	0.605192	0.396067	0.029809
			8	0.323607	0.129815	0.025093				8	0.461159	0.234633	0.021965
			9	0.235851	0.071359	0.015734				9	0.376452	0.158084	0.016368
			10	0.181248	0.043209	0.010359				10	0.318968	0.114324	0.012583
		8	8	0.512034	0.301553	0.039374			8	8	0.625768	0.419129	0.027543
			9	0.348680	0.146516	0.024938				9	0.485361	0.256504	0.020929
			10	0.259252	0.083423	0.016212				10	0.401088	0.176839	0.015967
		9	9	0.532454	0.320933	0.037426			9	9	0.643319	0.439457	0.025597
			10	0.371038	0.162289	0.024620				10	0.506430	0.276421	0.019950
		10	10	0.550389	0.338560	0.035632			10	10	0.658529	0.457572	0.023911

Table 2: Covariances of order statistics for $n = 2, 3, ..., 10$, $\delta = 0.5$ and $\xi = 1, 2$

δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
0.5	1	2	1	2	0.027778	0.010001	0.5	2	2	1	2	0.071111	0.014553
				3	0.008730	0.004138					3	0.032354	0.008586
				4	0.003608	0.001914					4	0.018197	0.005346
				5	0.001758	0.000992					5	0.011550	0.003578
				6	0.000957	0.000562					6	0.007933	0.002537
				7	0.000566	0.000341					7	0.005761	0.001882
				8	0.000356	0.000219					8	0.004361	0.001445
				9	0.000235	0.000147					9	0.003408	0.001141
				10	0.000161	0.000102					10	0.002732	0.000922
		3	1	3	0.015873	0.003883			3	1	3	0.052117	0.006505
				4	0.006118	0.002097					4	0.027582	0.004739
				5	0.002886	0.001148					5	0.017013	0.003351
				6	0.001543	0.000668					6	0.011489	0.002441
				7	0.000901	0.000412					7	0.008250	0.001839
				8	0.000561	0.000267					8	0.006194	0.001427
				9	0.000368	0.000180					9	0.004811	0.001135
				10	0.000251	0.000126					10	0.003838	0.000921

Table 2: Continued.

δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
			2	3	0.058730	0.015567				2	3	0.128862	0.018179
				4	0.021587	0.007801					4	0.065441	0.012284
				5	0.009879	0.004085					5	0.039322	0.008307
				6	0.005173	0.002310					6	0.026078	0.005876
				7	0.002973	0.001396					7	0.018478	0.004333
				8	0.001831	0.000891					8	0.013731	0.003308
				9	0.001190	0.000594					9	0.010579	0.002598
				10	0.000807	0.000411					10	0.008384	0.002088
		4	1	4	0.010476	0.001870			4	1	4	0.041590	0.003539
				5	0.004607	0.001196					5	0.024186	0.002935
				6	0.002382	0.000735					6	0.015880	0.002262
				7	0.001363	0.000466					7	0.011213	0.001752
				8	0.000838	0.000307					8	0.008324	0.001382
				9	0.000545	0.000210					9	0.006413	0.001111
				10	0.000369	0.000148					10	0.005084	0.000908
			2	4	0.036421	0.006913				2	4	0.097705	0.009158
				5	0.015607	0.004237					5	0.055513	0.007269
				6	0.007923	0.002534					6	0.035848	0.005440
				7	0.004472	0.001575					7	0.025001	0.004127
				8	0.002722	0.001022					8	0.018381	0.003203
				9	0.001753	0.000689					9	0.014053	0.002542
				10	0.001181	0.000481					10	0.011072	0.002059
			3	4	0.088456	0.018373				3	4	0.176151	0.018759
				5	0.036286	0.010462					5	0.096526	0.013850
				6	0.017917	0.005990					6	0.060923	0.009934
				7	0.009922	0.003619					7	0.041819	0.007327
				8	0.005954	0.002301					8	0.030392	0.005574
				9	0.003795	0.001529					9	0.023028	0.004358
				10	0.002535	0.001054					10	0.018015	0.003488
		5	1	5	0.007511	0.001028			5	1	5	0.034785	0.002169
				6	0.003627	0.000740					6	0.021599	0.001964
				7	0.002009	0.000496					7	0.014844	0.001611
				8	0.001212	0.000336					8	0.010843	0.001308
				9	0.000777	0.000233					9	0.008262	0.001069
				10	0.000522	0.000166					10	0.006499	0.000884
			2	5	0.025241	0.003633				2	5	0.079409	0.005367
				6	0.011993	0.002544					6	0.048563	0.004721
				7	0.006559	0.001672					7	0.032991	0.003793
				8	0.003918	0.001116					8	0.023878	0.003030
				9	0.002492	0.000765					9	0.018062	0.002446
				10	0.001664	0.000540					10	0.014122	0.002003

Table 2: Continued.

δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
			3	5	0.058009	0.008926				3	5	0.137098	0.010211
				6	0.026904	0.005994					6	0.082128	0.008615
				7	0.014463	0.003832					7	0.054979	0.006731
				8	0.008528	0.002507					8	0.039359	0.005271
				9	0.005372	0.001694					9	0.029521	0.004192
				10	0.003559	0.001181					10	0.022925	0.003392
			4	5	0.115995	0.019683				4	5	0.215710	0.018332
				6	0.051660	0.012288					6	0.125061	0.014412
				7	0.027063	0.007528					7	0.082007	0.010805
				8	0.015676	0.004789					8	0.057876	0.008235
				9	0.009747	0.003173					9	0.042955	0.006424
				10	0.006393	0.002180					10	0.033089	0.005124
		6	1	6	0.005684	0.000620			6	1	6	0.029980	0.001440
				7	0.002947	0.000487					7	0.019544	0.001390
				8	0.001722	0.000349					8	0.013913	0.001196
				9	0.001083	0.000249					9	0.010439	0.001006
				10	0.000718	0.000180					10	0.008125	0.000847
			2	6	0.018701	0.002126				2	6	0.067187	0.003459
				7	0.009586	0.001638					7	0.043326	0.003271
				8	0.005550	0.001156					8	0.030576	0.002769
				9	0.003465	0.000816					9	0.022779	0.002302
				10	0.002285	0.000585					10	0.017627	0.001918
			3	6	0.041676	0.004997				3	6	0.113165	0.006307
				7	0.021036	0.003746					7	0.071990	0.005802
				8	0.012036	0.002594					8	0.050282	0.004816
				9	0.007447	0.001804					9	0.037158	0.003945
				10	0.004874	0.001279					10	0.028566	0.003248
			4	6	0.079265	0.010203				4	6	0.171378	0.010538
				7	0.039110	0.007340					7	0.106985	0.009308
				8	0.022016	0.004945					8	0.073730	0.007521
				9	0.013458	0.003372					9	0.053944	0.006043
				10	0.008727	0.002357					10	0.041152	0.004905
			5	6	0.141262	0.020173				5	6	0.249456	0.017565
				7	0.067095	0.013505					7	0.151106	0.014474
				8	0.036865	0.008724					8	0.102175	0.011233
				9	0.022160	0.005787					9	0.073779	0.008790
				10	0.014195	0.003966					10	0.055740	0.007002

Table 2: Continued.

δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
		7	1	7	0.004470	0.000399			7	1	7	0.026389	0.001013
				8	0.002451	0.000335					8	0.017866	0.001026
				9	0.001495	0.000253					9	0.013080	0.000916
				10	0.000973	0.000188					10	0.010030	0.000794
			2	7	0.014494	0.001342				2	7	0.058373	0.002382
				8	0.007880	0.001111					8	0.039196	0.002376
				9	0.004773	0.000830					9	0.028502	0.002095
				10	0.003090	0.000612					10	0.021734	0.001798
			3	7	0.031674	0.003065				3	7	0.096746	0.004223
				8	0.017036	0.002490					8	0.064334	0.004131
				9	0.010230	0.001833					9	0.046420	0.003589
				10	0.006577	0.001337					10	0.035174	0.003044
			4	7	0.058559	0.005991				4	7	0.143309	0.006769
				8	0.031038	0.004740					8	0.094118	0.006449
				9	0.018431	0.003422					9	0.067269	0.005497
				10	0.011746	0.002462					10	0.050596	0.004597
			5	7	0.099657	0.010985				5	7	0.201510	0.010516
				8	0.051693	0.008341					8	0.130045	0.009626
				9	0.030227	0.005863					9	0.091801	0.007992
				10	0.019045	0.004137					10	0.068410	0.006559
			6	7	0.164416	0.020206				6	7	0.278715	0.016705
				8	0.082268	0.014289					8	0.174891	0.014281
				9	0.047015	0.009634					9	0.121288	0.011395
				10	0.029155	0.006616					10	0.089283	0.009113
		8	1	8	0.003617	0.000271			8	1	8	0.023594	0.000744
				9	0.002075	0.000240					9	0.016466	0.000783
				10	0.001311	0.000189					10	0.012335	0.000720
			2	8	0.011606	0.000896				2	8	0.051684	0.001722
				9	0.006615	0.000786					9	0.035837	0.001790
				10	0.004158	0.000614					10	0.026698	0.001630
			3	8	0.025021	0.002006				3	8	0.084684	0.002992
				9	0.014148	0.001734					9	0.058289	0.003067
				10	0.008834	0.001340					10	0.043164	0.002762
			4	8	0.045423	0.003813				4	8	0.123631	0.004668
				9	0.025422	0.003234					9	0.084338	0.004694
				10	0.015744	0.002464					10	0.062007	0.004166

Table 2: Continued.

δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	δ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
			5	8	0.075291	0.006697				5	8	0.170366	0.006964
				9	0.041552	0.005532					9	0.114878	0.006824
				10	0.025459	0.004136					10	0.083720	0.005945
			6	8	0.118997	0.011436				6	8	0.228260	0.010322
				9	0.064336	0.009073					9	0.151410	0.009724
				10	0.038844	0.006604					10	0.109066	0.008256
			7	8	0.185677	0.019979				7	8	0.304429	0.015850
				9	0.097007	0.014771					9	0.196672	0.013956
				10	0.057307	0.010318					10	0.139328	0.011394
		9	1	9	0.002994	0.000191			9	1	9	0.021352	0.000565
				10	0.001783	0.000177					10	0.015278	0.000613
			2	9	0.009527	0.000625				2	9	0.046418	0.001292
				10	0.005646	0.000574					10	0.033042	0.001391
			3	9	0.020337	0.001379				3	9	0.075405	0.002211
				10	0.011979	0.001252					10	0.053358	0.002345
			4	9	0.036452	0.002570				4	9	0.108949	0.003386
				10	0.021307	0.002301					10	0.076590	0.003557
			5	9	0.059395	0.004390				5	9	0.148136	0.004917
				10	0.034376	0.003859					10	0.103251	0.005050
			6	9	0.091576	0.007186				6	9	0.194801	0.007005
				10	0.052295	0.006154					10	0.134318	0.007030
			7	9	0.137247	0.011667				7	9	0.252222	0.010043
				10	0.076847	0.009597					10	0.171222	0.009687
			8	9	0.205262	0.019606				8	9	0.327281	0.015038
				10	0.111231	0.015038					10	0.216691	0.013568
		10	1	10	0.002522	0.000139			10	1	10	0.019510	0.000441
			2	10	0.007976	0.000452				2	10	0.042156	0.000998
			3	10	0.016897	0.000985				3	10	0.068024	0.001690
			4	10	0.030003	0.001809				4	10	0.097514	0.002546
			5	10	0.048301	0.003031				5	10	0.131332	0.003637
			6	10	0.073274	0.004828				6	10	0.170556	0.005040
			7	10	0.107274	0.007517				7	10	0.217004	0.006954
			8	10	0.154439	0.011750				8	10	0.273856	0.009728
			9	10	0.223369	0.019154				9	10	0.347783	0.014284

Table 3: Coefficient of the BLUEs of the p_u

δ	ξ	n	c	p_u, $u = 1, 2, \ldots, (m - c)$						
0.5	1	7	0	1.122123	-0.075374	-0.023245	-0.009345	-0.004594	-0.002852	-0.006712
			1	1.129701	-0.076824	-0.024211	-0.010194	-0.005477	-0.012996	
			2	1.141753	-0.07901	-0.025653	-0.011471	-0.025621		
		10	0	1.120081	-0.075032	-0.023505	-0.009452	-0.004447	-0.002353	-0.001384
				-0.000921	-0.000737	-0.002249				
			1	1.122372	-0.075425	-0.023748	-0.009646	-0.004623	-0.002525	-0.001566
				-0.001129	-0.003709					
			2	1.125214	-0.075894	-0.024035	-0.009874	-0.00483	-0.002731	-0.001783
				-0.006068						
			3	1.129322	-0.076545	-0.024429	-0.010186	-0.005113	-0.003011	-0.010038
			4	1.135917	-0.077546	-0.025026	-0.010658	-0.005544	-0.017143	
	2	7	0	1.129977	-0.025254	-0.016827	-0.013254	-0.012016	-0.012837	-0.049791
			1	1.162885	-0.029873	-0.021133	-0.017699	-0.017052	-0.077129	
			2	1.205879	-0.035752	-0.026621	-0.023376	-0.120132		
		10	0	1.095827	-0.018305	-0.011527	-0.008401	-0.006652	-0.005841	-0.005352
				-0.005636	-0.006409	-0.027705				
			1	1.111367	-0.019731	-0.013508	-0.009195	-0.008641	-0.007048	-0.007175
				-0.007635	-0.038435					
			2	1.128081	-0.021833	-0.014442	-0.011367	-0.009619	-0.008913	-0.008861
				-0.053046						
			3	1.148688	-0.023804	-0.016537	-0.013022	-0.011512	-0.010857	-0.072956
			4	1.176809	-0.026692	-0.018993	-0.015504	-0.013931	-0.101689	

Table 4: Coefficient of the BLUEs of the q_u

δ	ξ	m	c	q_u, $u = 1, 2, \ldots, (m-c)$						
0.5	1	7	0	-3.46878	0.549522	0.341474	0.287862	0.289566	0.344431	1.655924
			1	-5.33835	0.907121	0.579507	0.497268	0.507453	2.846996	
			2	-7.97863	1.386104	0.895646	0.776871	4.920006		
		10	0	-3.40004	0.483125	0.278286	0.213224	0.188383	0.181781	0.188651
				0.212461	0.271219	1.382916				
			1	-4.80901	0.725179	0.427867	0.332547	0.296316	0.287677	0.300279
				0.340695	2.098449					
			2	-6.41672	0.990501	0.590117	0.461615	0.413331	0.403162	0.423261
				3.134731						
			3	-8.53949	1.326667	0.793374	0.622812	0.559831	0.548687	4.688121
			4	-11.6192	1.793964	1.072399	0.843379	0.760811	7.148644	
	2	7	0	-2.46947	0.220001	0.198251	0.199872	0.222026	0.280561	1.348755
			1	-3.36093	0.345122	0.314918	0.320282	0.358451	2.022157	
			2	-4.48814	0.499256	0.458759	0.469136	3.060991		
		10	0	-2.3293	0.155376	0.131581	0.126299	0.121686	0.131588	0.141512
				0.168971	0.222404	1.129882				
			1	-2.96309	0.213516	0.212391	0.158725	0.202792	0.180793	0.215848
				0.250522	1.528504					
			2	-3.62776	0.297134	0.249521	0.245132	0.241661	0.254961	0.282914
				2.056436						
			3	-4.42664	0.373541	0.330768	0.309272	0.315052	0.330328	2.767681
			4	-5.49345	0.483106	0.423914	0.403443	0.406811	3.776179	

Table 5: Variances and covariance of the BLUEs

δ	ξ	m	c	$var(a^*)$	$var(b^*)$	$cov(a^*, b^*)$
0.5	1	7	0	0.000251	0.167441	-0.000853
			1	0.000252	0.284615	-0.001328
			2	0.000255	0.430011	-0.001992
		10	0	0.000071	0.109877	-0.000236
			1	0.000072	0.171915	-0.000337
			2	0.000072	0.236574	-0.000451
			3	0.000072	0.313967	-0.000601
			4	0.000073	0.414729	-0.000817
	2	7	0	0.002146	0.086711	-0.005036
			1	0.002217	0.138541	-0.006951
			2	0.002309	0.201735	-0.009361
		10	0	0.000991	0.054401	-0.002226
			1	0.001006	0.080332	-0.002861
			2	0.001022	0.106462	-0.003519
			3	0.001043	0.137329	-0.004315
			4	0.001071	0.177438	-0.005372