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The security mortgage valuation in a stochastic perspective

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The reverse mortgage market has been expanding rapidly in developed economies in recent years. Reverse mortgages provide an alternative source of funding for retirement income and health care costs. Increase in life expectancies and decrease in the real income at retirement continue to worry those who are retired or close to retirement. Therefore, financial products that help to alleviate the "risk of living longer" continue to be attractive among the retirees. Reverse mortgage contracts involve a range of risks from the insurer's perspective. When the outstanding balance exceeds the housing value before the loan is settled, the insurer suffers an exposure to crossover risk induced by three risk factors: interest rates, house prices and mortality rates. We analyse the combined impact of these risks on pricing and the risk profile of reverse mortgage loans in a stochastic interest-mortality-house pricing model. Our results show that pricing of reverse mortgages loans does not accurately assess the risks underwritten by reverse mortgages lenders.In particular, it fails to take into account mortality improvements substantially underestimating the longevity risk involved in reverse mortgage loans.

keywords: Equity release products, reverse mortgage, stochastic mortality, CIR model.

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1 Introduction

Demographic ageing represents one of the most serious challenge for developed and developing countries. The trends of mortality improvement show that in many countries around the world, life expectancy from birth is well over 80 years. Governments and industries seek to decrease their financial burden by deferring the retirement age or reducing the benefits people receive. Empirical research shows that retired people often have plenty of equity locked in their homes but very few liquid assets to rely on to support their daily needs. For this reason, we often hear the phrase "house rich and cash poor" to refer to the increasing number of elderly who hold a substantial proportion of their assets in home equity. The financial industry is again starting to tap into this market because of the fact that reverse mortgages might provide a more practical solution. Reverse mortgages are financial contracts that allow retirees to convert their home equities into either a lump sum or an annuity, but still maintain ownership and residence until they die, sell or vacate their home to live elsewhere. Loans made through a reverse mortgage accrue with interest and are settled only upon the death of the borrower, sale of the property or the vacation of its residents. There are no repayments made during the course of the loan, and no assets other than the home may be attached to debt repayment. If, at time of settlement, the loan accrued with interest is larger than the sale price of the property, then the provider (or lender), usually a bank or an insurance company, recover only up to the sale price of the property.
In Italy, reverse mortgages, known as PIV, were established in 2005 and then modified in 2015. Today, although the demand for them remains to be scant, there are early indications of a rapid growth for these products. The reverse mortgage products involve a range of risks from the provider's perspective. The outstanding balance usually accumulates at a faster rate than the appreciating rate of the housing value, therefore, if the outstanding balance exceeds the housing value before the loan is settled, the lender starts to incur in a loss. The crossover risk is crucial for managing reverse mortgages effectively. It can be induced by three risk factors: interest rates, housing prices, and mortality rates. Improvements in mortality will delay the settlement of the reverse mortgage loan and will therefore increase the chances oh hitting the crossover mark. An high interest rate will increase the rate at which the loan balance will accrue, and will therefore possibly hit the crossover earlier than expected. Finally, a depressed real estate market will worsen the value of the house.
In this field Wang et al. (2012) proposed a securitization method to hedge longevity risk inherent in reverse mortgage products. Huang et al. (2011), using a three dimensional lattice method, numerically calculate fair coupon rates for crossover bonds. Shao et al. (2015) study the large idiosyncratic component in the house price risk. Tsay et al. (2014) allow the house prices and interest rates to be stochastic with a deterministic distribution of termination time. Dowd et al. (2019) outline the valuation process for a negative equity guarantee.
In this paper, referring to Italian case, we consider the effect of interest, mortality and stock prices stochastic dynamics in order to investigate the actuarial balance of the contract. The paper is organized as follows. Section 2 examines the reverse mortgage

contract and present the stochastic model for interest rates, mortality rates and house prices. In section 3 we develop a model to capture the stochastic dynamic of the contract. Section 4 illustrate the results and concludes.

2 The Reverse mortgage contract

In this section we model an Italian Reverse mortgage loan introducing the notation to be used throughout of the paper.
Let us consider a retired individual aged x who takes out a reverse mortgage loan for a value M_0, against his house currently valued I_0. If, at time t, the loan amount is M_t and the house price is I_t, then, by definition, the value of a reverse mortgage loan is :

$$Y_t = \min(M_t, I_t) \tag{1}$$

On the basis of put decomposition we can rewrite (1) as follows:

$$Y_t = M_t - \max(M_t - I_t, 0) \tag{2}$$

Therefore, entering into a reverse mortgage contract, for the borrower , is equivalent to taking out a loan without a real estate guarantee and buyng a put option written on the value of the mortgaged property, with random exercise price and random exercise time. In our case, by definition, the duration of the contract is equal to the remaining lifetime of the insured. Let us suppose that the loan amount M_t accumulates at a risk free rate r_t and the house price I_t appreciates at a rate δ_t, then:

$$M_t = M_0 e^{\int_0^{\tilde{s}} r_t dt} \tag{3}$$

$$I_t = I_0 e^{\int_0^{\tilde{s}} \delta_t dt} \tag{4}$$

$$\begin{aligned} Y_t &= \min\left(M_0 e^{\int_0^{\tilde{s}} r_t dt}, I_0 e^{\int_0^{\tilde{s}} \delta_t dt} \right) \\ &= M_0 e^{\int_0^{\tilde{s}} r_t dt} - \max\left(M_0 e^{\int_0^{\tilde{s}} r_t dt} - I_0 e^{\int_0^{\tilde{s}} \delta_t dt}, 0 \right) \end{aligned} \tag{5}$$

where $\tilde{s}$ is the remaining lifetime of the insured. For practical purposes, it must be considered that only a percentage α of the initial house value I_0 is granted. The coefficient α depends on the age x of the borrower at issue.
Looking at the contract from the lender's point of view, the Net Asset Value (NAV) at time 0 is given by the borrower's liability at maturity determined at issue minus the initial capital paid out:

$$\begin{aligned} NAV(0) &= V(0, Y_s) - \alpha I_0 = \\ &= \sum_{k=1}^{\omega} V(0, Y_k)_{k-1/1} q_x - \alpha I_0 \end{aligned} \tag{6}$$

where

$$Y_s = \begin{cases} \min(M_k, I_k) & with \quad probability \quad {}_{k-1/1}q_x \\ 0 & with \quad probability \quad 1 -_{k-1/1} q_x, \end{cases} \tag{7}$$

where ω is the extreme age and ${}_{k-1/1}q_x$ is the deferred probability of death of an individual aged x.
In the case of put decomposition we have:

$$\begin{aligned} NAV(0) &= V(0, M_{\tilde{s}}) - V(0, \max(M_{\tilde{s}} - I_{\tilde{s}}, 0)) - \alpha I_0 = \\ &= \sum_{k=1}^{\omega} (M_k v_k^* - P_k)_{k-1/1}q_x - \alpha I_0 \end{aligned} \tag{8}$$

Where

$$M_k = \alpha I_0 e^{\int_0^k r_t dt}$$

$$v_k^* = e^{-\int_0^k \mu_t dt}$$

$$P_k = V(0, \max(M_k - I_k, 0))$$

with μ representing the cost of capital rate.

2.1 Interest rate process

The interest rate dynamics is described by means of the diffusion process.

$$dr_t = f^r(r_t, t)dt + l^r(r_t, t)dZ_t^r \tag{9}$$

where $f^r(r_t, t)$ is the drift of the process, $l^r(r_t, t)$ is the diffusion coefficient and dZ_t^r is a standard brownian motion.
In particular, in the Cox Ingersoll and Ross model (Cox et al., 1985) the drift and diffusion coefficient are defined respectively as:

$$f^r(r_t, t) = \kappa(\theta - r_t)$$

$$l^r(r_t, t) = \sigma_r \sqrt{r_t}$$

Where κ is the mean reverting coefficient, θ is the long term period normal rate and σ_r is the spot volatility rate.

2.2 Housing price model

Mortgage valuation tipically relies on the assumption that housing prices follow a stochastic geometric brownian motion process. The diffusion process for this dynamics is given by the stochastic differential equation

$$dS_t = f^s(S_t, t)dt + g^s(S_t, t)dZ_t^S \tag{10}$$

As a Black-Scholes type model (Black and Scholes , 1973) we assume

$$f^s(S_t, t) = \mu_s S_t$$

$$g^s(S_t, t) = \sigma_s S_t$$

In our case,the valuation of P_k in (8) is given by:

$$\begin{aligned} P_k &= V(0, \max(M_k - I_k, 0)) \\ &= M_n e^{-rn} N(-d_2) - I_n e^{-qn} N(-d_1) \end{aligned} \tag{11}$$

with

$$d_1 = \frac{log \frac{I_n}{M_n} + (r - q + \frac{\sigma^2}{2}(T - t))}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

2.3 The mortality model

The mortality rate dynamics is given by:

$$d\mu_{x+t} = f^\mu(\mu_{x+t}, t)dt + h^\mu(\mu_{x+t}, t)dZ_t^\mu \tag{12}$$

As a CIR mortality model is assumed, you get:

$$f^\mu(\mu_{x+t}, t) = \kappa(\gamma - \mu_{x+t})$$

$$h^\mu(\mu_{x+t}, t) = \sigma_\mu\sqrt{\mu_{x+t}}$$

κ and $sigma$ are positive constants, γ is the long term mean and Z_t is a Standard Brownian Motion.
The CIR mortality model is a widely used stochastic mortality model (Cox et al., 1985) that describes the evolution in time of mortality. This model has the property that mortality rates are continuous and remain positive.

3 The CIR centered model

We now introduce the centered version of the model. Since both the interest and the mortality process are modeled through a Cir type process, we show the procedure to obtain the survival probabilities rate. The same procedure is applied to interest rates.
Let us consider the shifted $\mu^*_{x+t} = \mu_{x+t} - \gamma$. The process is then centred around γ and the long term mean converges almost everywhere to zero:

$$d\mu^*_{x+t} = \kappa\mu^*_{x+t}dt + \sigma\sqrt{\mu^*_{x+t} + \gamma}dB_t \tag{13}$$

with initial condition given by the known value of μ_{x+t}. Its solution is given by

$$E[\mu^*_{x+t}] = e^{-\kappa t}\mu^*_{x+0} \tag{14}$$

$$cov(\mu^*_{x+t}, \mu^*_{x+s}) = \sigma^2 \frac{e^{-\kappa t} - e^{-\kappa(s+t)}}{\kappa} \mu^*_{x+0} + \sigma^2 \frac{e^{-\kappa(t-s)} - e^{-\kappa(s+t)}}{2\kappa} \gamma, s \leq t \tag{15}$$

$$\lim_{t \to \infty} Var[\mu^*_{x+t}] = \frac{\gamma\sigma^2}{2\kappa} \tag{16}$$

3.1 Parameter estimation procedure

Estimating the parameters of the stochastic mortality model requires the discrete representation of the model. To this aim, we refer to the covariance equivalence principle (Deelstra and Parker , 1995) which requires that the expected values and the stationary variances of the continuous and discrete processes to be equal. The discrete model representation is given by the following equation:

$$\mu^*_{x+t} = \phi\mu_{x+t-1} + \sigma_a \sqrt{\frac{2\phi}{1+\phi}\mu_{x+t-1} + \gamma a_t} \tag{17}$$

The expected value, the covariance and stationary variance functions of the previous equation are:

$$E[\mu^*_{x+t}] = \phi^t \mu^*_{x+0} \tag{18}$$

$$cov(\mu^*_{x+t}, \mu^*_{x+s}) = 2\phi^t \sigma_a^2 \mu^*_{x+0} \frac{1-\phi^s}{1-\phi^2} + \phi^{t-s}\sigma_a^2 \gamma \frac{1-\phi^{2s}}{1-\phi^2} \tag{19}$$

$$\lim_{t \to \infty} Var[\mu^*_{x+t}] = \frac{\sigma_a^2 \gamma}{1-\phi^2} \tag{20}$$

The estimation procedure starts by finding the value of ϕ that minimizes the residual sum of squares function:

$$RSS = \sum_{t=1}^{N} \frac{(\mu^*_{x+t} - \phi\mu^*_{x+t-1})^2}{\frac{2\phi}{1+\phi}\mu^*_{x+t} + \gamma} \tag{21}$$

The least squares estimate of σ_a^2 is given by RSS/N-1. Finally the continuous model parameters are obtained by means of the parametric relationships between continuous and discrete models, derived by applying the covariance equivalence principle:

$$\phi = e^{-\kappa} \tag{22}$$

$$\sigma_a^2 = \sigma^2 \frac{1 - e^{-2\kappa}}{2\kappa} \tag{23}$$

At this point, by the Pitman and Yor formula (Pitman et al., 1982), we can compute the survival probability

$$_kp_x(t) = E[e^{-\int_0^k \mu_{x+s} ds}] = \frac{exp(-\frac{x}{\sigma^2} w \frac{1+\frac{k}{w} coth(wk/2)}{coth(wk/2)} + (k/w)}{(coth(wt/2) + (k/w) sinhcoth(wk/2))^{\frac{2k\gamma}{\sigma^2}}} \tag{24}$$

where $x = \mu_0$ e $w = \sqrt{k^2 + s\sigma^2}$.

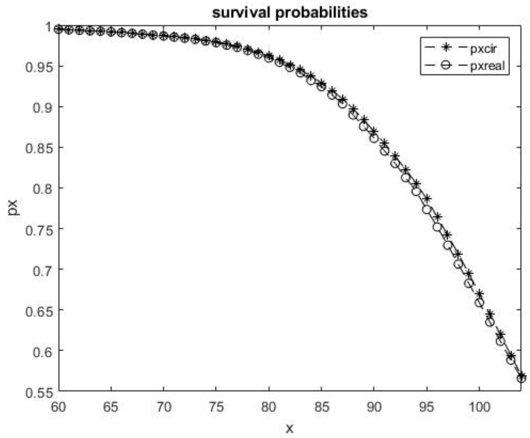

Figure 1: Annual survival probabilities $p_x(t)$ with $x \in (60, 105)$ for each calendar year t ranging from 1968 to 2013. Comparison between CIR model and real data.

4 Numerical results

Applying the described estimation procedure the significant parameters of the mortality –CIR model are obtained. Our data set relates to the Italian population with age specific death counts from ages 60 to 105 over the period 1968-2013. We refer to the class of the forward mortality models(data source Human mortality database. www.mortality.org). These models study changes in the mortality rate curve for a specific age cohort and capture dynamics of eache age cohort for all the ages grater than x in the year t (for example age x in the year t, $x+1$ in the year $t+1$ and so on). The parameters obtained for k and σ^2 are $k = 0.0010005$ and $\sigma^2 = 0.036714863$. Figure 1 shows the comparison between the annual survival probabilities obtained by means of the CIR model and the corresponding probabilities of the Italian population.
The interest CIR model follows the same logic of the previous one. This time, interest rate date for the IRS 5-25 curves for the period ranging from January 2001 to December 2019 were considered. Figure 2 shows the results.

The purpose of this application is to study the behavior of the NAV, defined by (5), and to investigate the behavior of the interest rate risk and the mortality risk. In our analysis we consider a borrower of initial age 60 owner of a property worth 100.000,00. Then, we let the initial age gradually vary. Given the initial age, the residual life span is estimated using the mortality CIR model. The annual collection rate for the lender is given by the IRS rate estimated by the interest CIR model for the corresponding maturity. In order to assess the put option price, we consider a free risk rate given by the IRS rate, a dividend rate equal to 0, and an annual volatility of the underlying of 10%.
Figure 3 shows the behavior of the NAV with an entry age x=60. In the first plot NAV values are assessed as the contractual rate changes. In this case, the residual

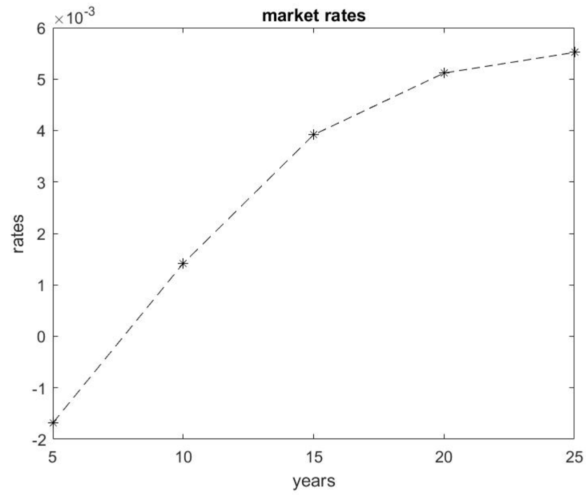

Figure 2: CIR interest rates. $\phi = 0.999$ $\sigma = 0.0007$

lifetime of the borrower is set at 26 years. The discount rate is the 25-year IRS rate while the percentage of the financed value is 20%. We can observe that the rate that makes the NAV equal to 0, is 3.145%. For higher rates, the lender runs the request that the mortgaged property does not cover the amount of the borrowed capital. The second plot studies the behavior of the NAV as the residual life varies. In this case, the considered contractual rate is the one that makes the NAV equal to zero. The other parameters remain unchanged. It can be noted that for life spans higher than 26 years, the lender runs the request that the mortgaged property does not cover the amount of the borrowed capital. This risk is quite accentuated and difficult to control when subscribing. In Figures 4, 5, 6, 7 the computation of NAV is repetead referring to ages of entry that gradually increase. The aim is the same, that is to evaluate investment risk and mortality risk. Obviously, the percentage of the property value financing varies. The discounting rate varies too. The results show a trend similar to that discussed for x = 60. In general, the results show that the products currently sold on the Italian market expose the lender to considerable financial and demographic risk.

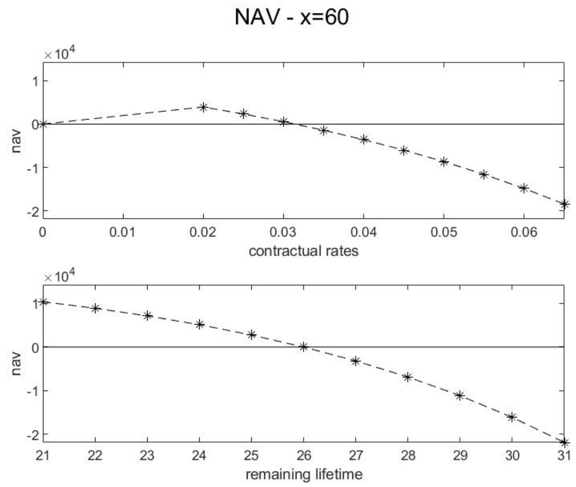

Figure 3: NAV as function of the contractual rate and residual lifetime fixed at 26 (first plot). NAV as function of the residual lifetime and contractual rate equal to 3.145% (second plot). Entry age x=60, $\alpha = 20\%$

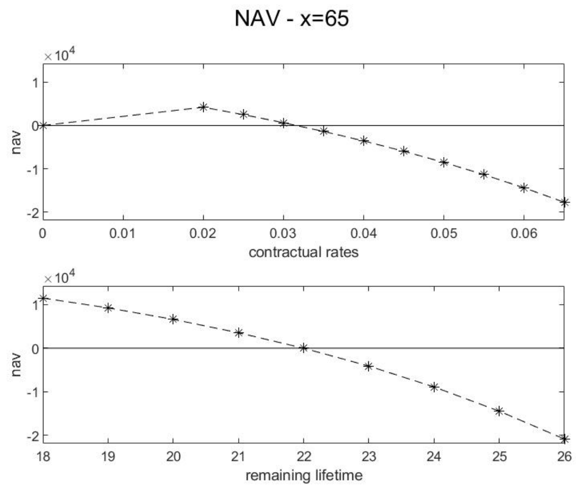

Figure 4: NAV as function of the contractual rate and residual lifetime fixed at 22 (first plot). NAV as function of the residual lifetime and contractual rate equal to 3.162% (second plot). Entry age x=65, $\alpha = 25\%$

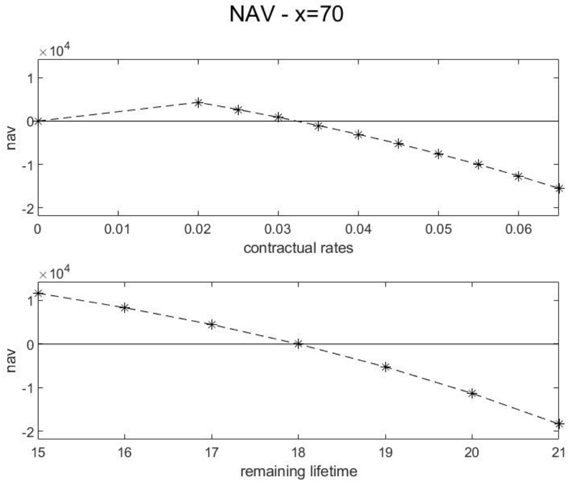

Figure 5: NAV as function of the contractual rate and residual lifetime fixed at 18 (first plot). NAV as function of the residual lifetime and contractual rate equal to 3.235% (second plot). Entry age x=70, $\alpha = 30\%$

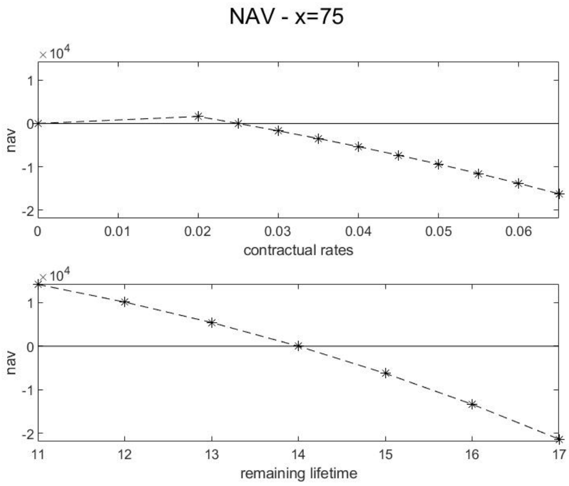

Figure 6: NAV as function of the contractual rate and residual lifetime fixed at 14 (first plot). NAV as function of the residual lifetime and contractual rate equal to 3.918% (second plot). Entry age x=75, $\alpha = 30\%$

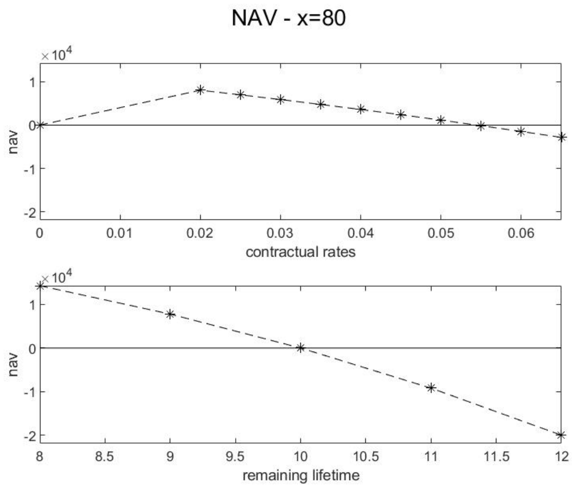

Figure 7: NAV as function of the contractual rate and residual lifetime fixed at 10 (first plot). NAV as function of the residual lifetime and contractual rate equal to 5.445% (second plot). Entry age x=80, $\alpha = 40\%$

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