Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
`http://siba-ese.unisalento.it/index.php/ejasa/index`
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v13n1p31

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Published: 02 May 2020

Electronic Journal of Applied Statistical Analysis
Vol. 13, Issue 01, May 2020, 31-46
DOI: 10.1285/i20705948v13n1p31

Statistical testing for the performance of lifetime index of transformed Rayleigh products under progressively type II right censored samples

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Published: 02 May 2020

In practice, process capability indices (PCIs) are widely used in the field of quality control. The lifetime performance index (C_L) is used to measure process potential and performance, where L is the lower specification limit. In this paper, we apply data transformation technology to construct a maximum likelihood estimator (MLE) of C_L under the two-parameter Raileigh distribution based on the progressively type II right censored sample. The MLE of C_L is then utilized to develop a hypothesis testing procedure. Finally, we give the Monte Carlo power simulation to assess the behavior of the lifetime perform index.

keywords: Process capability index, The lifetime performance index, Progressive type II right censored sample, Maximum likelihood estimator, Two-parameter Rayleigh Distribution.

1 Introduction

In the manufacturing industry, process capability indices (PCIs) are utilized to assess whether product quality meets the required level. Under the assumption of normality of data in the process capability analysis, Montgomery (1985) proposed the process capability index C_L or (C_{PL}) for evaluating the lifetime performance of electronic components, where L is the lower specification limit. Nevertheless, many researchers noted

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ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

that the lifetime model of many products frequently possesses a non-normal distribution including exponential, gamma, Weibull, Lomax, Burr XII distribution, etc.

Recently there have been a number of works on the generalizations and modifications of classical PCIs using the progressive Type-II censoring scheme to handle non-normal quality data, see for example Lee et al. (2009), Amani S Alghamdi (2019), Lee et al. (2010), Lee et al. (2012b), Lee et al. (2012a), Tong et al. (2002), EL-Sagheer (2017), Hong et al. (2009), Lee et al. (2011a), Viveros and Balakrishnan (1994), Dey et al. (2016), Montgomery (1985), Gunasekera and Wijekularathna (2018), Abbas Pak (2018), Wijekularathna and Subedi (2019). Also, Lee et al. (2011b) have constructed a Bayesian estimator of C_L under the assumption of one-parameter Rayleigh distribution and used the Bayesian estimator to develop a credible interval for C_L.

In this paper, we develop the statistical inference for C_L under the two-parameter Rayleigh distribution with the progressive type II right censored sample. The Rayleigh distribution has been recognized as a useful model for the analysis of lifetime data because it has some nice relations with some of the well known distributions like Weibull, chi-square or extreme value distributions. An important characteristic of the Rayleigh distribution is that its hazard function is an increasing function of time.

In life testing experiments, it can be difficult for experimenters to observe the lifetimes of all products tested due to time or other resources restrictions. Therefore, censored samples are commonly used in practice. In this study, we consider the progressive type-II censoring scheme in which we only observe the failure times. The m ordered observed failure times are denoted by $X_{1,n} \leq X_{2,n} \leq ..., \leq X_{m,n}$, and the number of surviving units removed at each failure time stage is denoted by $R_1, R_2, ..., R_m$. It is clear that $R_m = n - \sum_{j=1}^{m-1} R_j - m$ and $0 \leq R_i \leq n - \sum_{j=1}^{i-1} R_j - i$ for $i = 2, 3, ..., m-1$.

Wijekularathna and Subedi (2019) transformed the two-parameter Rayleigh distribution to one-parameter Rayleigh distribution, then constructed the maximum likelihood estimate of C_L under one-parameter Rayleigh distribution. In this study, we transform the two-parameter Rayleigh distribution to the exponential distribution and then use the exponential maximum likelihood estimate of C_L to conduct the hypothesis testing for C_L. The exponential transformation makes the computing procedure easier. In section 2 we give introduction on the lifetime performance index and the conform rate. In section 3 and 4 we construct MLE of C_L with the progressively type-II censored sample, and then develop the hypothesis testing procedures for C_L. In section 5 we perform Monte Carlo simulation to obtain the power of the statistical test. Finally we end the paper with some concluding remarks.

2 The lifetime performance index and the conforming rate

Suppose that the lifetime (X) of products (or items) can be modeled by a two-parameter Rayleigh Distribution with the probability density function (p.d.f) and cumulative distribution function (c.d.f) as

$$f(x; \lambda, \mu) = 2\lambda(x-\mu)e^{-\lambda(x-\mu)^2} \quad x > \mu, \quad \lambda > 0, \quad \mu > 0, \tag{1}$$

and

$$F(x; \lambda, \mu) = 1 - e^{-\lambda(x-\mu)^2} \quad x > \mu, \quad \lambda > 0, \quad \mu > 0, \tag{2}$$

By using the transformation $Y = (X - \mu)^2$, $(X > \mu)$, (see *Olive* (2014)) and the distribution of Y has exponential distribution with one parameter λ with the probability density function and cumulative distribution function as

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0, \quad \lambda > 0, \tag{3}$$

and

$$F_Y(y) = 1 - e^{-\lambda y}, \quad y \geq 0, \quad \lambda > 0, \tag{4}$$

respectively and the failure rate function $r(y)$ is defined by:

$$r(y) = \frac{f_Y(y)}{1 - F_Y(y)} = \frac{\lambda e^{-\lambda y}}{1 - [1 - e^{-\lambda y}]} = \lambda, \quad \lambda > 0. \tag{5}$$

Hence, if $X_{1,n} \leq X_{2,n} \leq ... \leq X_{m,n}$ is the right type II censored sample, then the new lifetimes $Y_{i,n} = (X_{i,n} - \mu)^2$, $\mu > 0 \ \forall \ i = 1, 2, ...m$ can be treated as the censored sample from the one parameter exponential distribution. Since the data transformation $Y = (X - \mu)^2$, $\mu > 0$ for $X > \mu$ is one to one and strictly increasing function, then both data set X and Y have the same effects in assessing the lifetime performance of products. For simplicity we will use one-parameter exponential distribution to derive the following lifetime performance index C_L and the conforming rate P_r based on data set Y.

Montgomery (1985) developed a capability index (the lifetime performance index) C_L to measure the lifetime larger-the-better type quality characteristics derived as follows:

$$C_L = \frac{\mu - L}{\sigma} = \frac{\frac{1}{\lambda} - L}{\frac{1}{\lambda}} = 1 - \lambda L; \quad C_L < 1, \tag{6}$$

where the process mean $\mu = E(Y) = \frac{1}{\lambda}$, and the process standard deviation $\sigma = \sqrt{var(Y)} = \frac{1}{\lambda}$, and L is the lower specification limit.

If the new lifetime of a product Y in which $Y = (X - \mu)^2$ exceeds the lower specification unit (*i.e.* $Y \geq L$), then the product is labeled as a conforming product. Otherwise, the product is labeled as a non-conforming product. The ratio of conforming products is known as the conforming rate and can be defined as

$$P_r = P(Y \geq L) = \int_L^\infty \lambda e^{-\lambda y} dy = e^{-\lambda L} = e^{C_L - 1}, \quad -\infty < C_L < 1. \tag{7}$$

It is noticed that there exists a strictly positive relationship between the conforming rate P_r and the lifetime performance index C_L, which implies that the larger the index value C_L, the larger conforming rate P_r. Moreover, this one-to-one mathematical relationship between C_L and P_r tells that lifetime performance index can be a flexible and effective tool, not only evaluating product quality, but also for estimating the conforming rate P_r. Montgomery (1985) suggested that the sample size must be large to accurately estimate the conforming rate. However, a large sample size is usually not practical due to cost and time or other restrictions. Therefore, alternately we can evaluate the conforming rate by evaluating the lifetime performance index.

We list various C_L values and the corresponding conforming rates P_r in Table 1. The clear positive relationship between the conforming rate P_r and the lifetime performance index C_L emerges from the results. For the C_L values which are not listed in Table 1, the conforming rate P_r can be calculated by by dividing the number of conforming products by the total number of products sampled.

Table 1: The lifetime performance index vs the conforming rate

C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.00000	-4.50	0.00409	0.10	0.40657
-9.00	0.00005	-4.25	0.00525	0.15	0.42741
-8.75	0.00006	-4.00	0.00674	0.20	0.44933
-8.50	0.00007	-3.75	0.00865	0.25	0.47237
-8.25	0.00010	-3.50	0.01111	0.30	0.49659
-8.00	0.00012	-3.25	0.01426	0.35	0.52205
-7.75	0.00016	-3.00	0.01832	0.40	0.54881
-7.50	0.00020	-2.75	0.02352	0.45	0.57695
-7.25	0.00026	-2.50	0.03020	0.50	0.60653
-7.00	0.00034	-2.25	0.03877	0.55	0.63763
-6.75	0.00043	-2.00	0.04979	0.60	0.67032
-6.50	0.00055	-1.75	0.06393	0.65	0.70469
-6.25	0.00071	-1.50	0.08208	0.70	0.74082
-6.00	0.00091	-1.25	0.10540	0.75	0.77880
-5.75	0.00117	-1.00	0.13534	0.80	0.81873
-5.50	0.00150	-0.75	0.17377	0.85	0.86071
-5.25	0.00193	-0.50	0.22313	0.90	0.90484
-5.00	0.00248	-0.25	0.28650	0.95	0.95123
-4.75	0.00318	0.00	0.36788	1.00	1.00000

3 MLE of the lifetime performance index

Let $X_{1,n} \leq X_{2,n} \leq ... \leq X_{m,n}$ from a progressive type II censoring scheme $R = (R_1, R_2, ..., R_m)$, the likelihood function can be written as

$$L(\lambda, \mu) = A \prod_{i=1}^{m} f_X(x_{i,n}; \lambda, \mu)[1 - F_X(x_{i,n}; \lambda, \mu)]^{R_i}, \tag{8}$$

where $A = n(n - R_1 - 1)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1)$, and $f_X(x_{i,n})$, $F_X(x_{i,n})$ are the *p.d.f* and *c.d.f* of X defined before as (1) and (2) respectively. So, the likelihood function is given by

$$\begin{aligned} L(\lambda, \mu) &= A \prod_{i=1}^{m} (2\lambda)(x_{i,n} - \mu) e^{-\lambda(x_{i,n}-\mu)^2} [1 - 1 + e^{-\lambda(x_{i,n}-\mu)^2}]^{R_i} \\ &= A(2\lambda)^m \prod_{i=1}^{m} (x_{i,n} - \mu) exp[-\lambda(R_i + 1)(x_{i,n} - \mu)^2]. \end{aligned} \tag{9}$$

Ignoring the additive constant, the log-likelihood function can be written as

$$Ln\ L(\lambda, \mu) = mln(\lambda) + \sum_{i=1}^{m} ln(x_{i,n} - \mu) - \lambda \sum_{i=1}^{m} (R_i + 1)(x_{i,n} - \mu)^2. \tag{10}$$

Assuming that μ is given, the maximum likelihood estimate of λ can be derived by solving the equation,

$$\frac{d\ Ln\ L(\lambda, \mu)}{d\ \lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} (R_i + 1)(x_{i,n} - \mu)^2 = 0. \tag{11}$$

Hence, the MLE of λ is given by,

$$\hat{\lambda} = \frac{m}{\sum_{i=1}^{m} (R_i + 1)(x_{i,n} - \mu)^2}, \tag{12}$$

By using the invariance of MLE, the MLE of C_L can be written as

$$\begin{aligned} \hat{C}_L = 1 - \hat{\lambda} L &= 1 - \frac{mL}{\sum_{i=1}^{m} (R_i + 1)(x_{i,n} - \mu)^2} \\ &= 1 - \frac{mL}{\sum_{i=1}^{m} (R_i + 1) y_{i,n}}, \end{aligned} \tag{13}$$

where $y_{i,n} = (x_{i,n} - \mu)^2$.

Let $W = \sum_{i=1}^{m} (1 + R_i) y_{i,n}$, then the MLE of C_L can be written as $\hat{C}_L = 1 - \frac{mL}{W}$. Viveros and Balakrishnan (1994) shows that $2\lambda W \sim \chi^2_{(2m)}$. Hence, the expectation of $\hat{C}_L$ can be derived as follows:

$$
\begin{aligned}
E(\hat{C}_L) &= E(1 - \frac{mL}{W}) \\
&= 1 - mLE(\frac{1}{W}) \\
&= 1 - 2\lambda mLE(\frac{1}{2\lambda W}),
\end{aligned}
\tag{14}
$$

where

$$
E(\frac{1}{W}) = \frac{\Gamma(m-1)}{2\Gamma(m)} = \frac{\Gamma(m-1)}{2(m-1)\Gamma(m-1)} = \frac{1}{2(m-1)}. \tag{15}
$$

Thus,

$$
E(\hat{C}_L) = 1 - \frac{2\lambda mL}{2(m-1)} = 1 - \frac{\lambda mL}{m-1}. \tag{16}
$$

But $E(\hat{C}_L) \neq C_L$, where $C_L = 1 - \lambda L$. Hence, the MLE $\hat{C}_L$ is not an unbiased estimator of C_L. But when $m \to \infty$, $E(\hat{C}_L) \to C_L$, so the MLE $\hat{C}_L$ is asymptotically unbiased estimator. Moreover, we also can show that the MLE $\hat{C}_L$ is consistent for C_L when $m \to \infty$ (see W.-C, Lee 2009).

4 Testing procedure for the lifetime performance index

In this section, we construct a statistical testing procedure to assess whether the lifetime performance index adheres to the required level. The one-sided hypothesis testing for C_L is obtained by using the pivotal quantity $2\lambda W$. Assuming that the required index value of the lifetime performance is larger than c, where c is the target value, the null hypothesis H_0 and the alternative hypothesis H_1 are given as:

$$
H_0 : C_L \leq c \qquad VS \qquad H_1 : C_L > c. \tag{17}
$$

The MLE $\hat{C}_L$ of C_L is used as the test statistic, so the rejection region can be expressed as $\{\hat{C}_L > C_0\}$, where the critical value C_0 for the given significance level α can be calculated as follows:

$$
\begin{aligned}
& P(\hat{C}_L > C_0 \mid C_L \leq c) \leq \alpha, \\
& \Rightarrow P(1 - \frac{mL}{W} > C_0 \mid 1 - \lambda L \leq c) \leq \alpha, \\
& \Rightarrow P(W > \frac{mL}{1 - C_0} \mid \lambda = \frac{1-c}{L}) = \alpha, \\
& \Rightarrow P(2\lambda W \leq \frac{2m(1-c)}{1 - C_0}) = 1 - \alpha,
\end{aligned}
\tag{18}
$$

From equation (18), utilizing the fact $2\lambda W \sim \chi^2_{2m}$,then we have

$$\frac{2m(1-c)}{1-C_0} = CHIINV(1-\alpha,\ 2m). \tag{19}$$

Thus, the critical value C_0 can be derived as

$$C_0 = 1 - \frac{2m(1-c)}{CHIINV(1-\alpha,\ 2m)}, \tag{20}$$

where c, m and α denote the target values, number of observed failures before termination and significance level respectively. Tables 2 and 3 list the critical values C_0 for m=2(1)50 and $c = 0.1(0.1)0.9$ at $\alpha = 0.01$ and at $\alpha = 0.05$, respectively.

Table 2: Critical value C_0 for m=2(1)50 and $c = 0.1(0.1)0.9$ at $\alpha = 0.01$

m	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5	c=0.6	c=0.7	c=0.8	c=0.9
1	0.8046	0.8263	0.8480	0.8697	0.8914	0.9131	0.9349	0.9566	0.9783
2	0.7288	0.7590	0.7891	0.8192	0.8494	0.8795	0.9096	0.9397	0.9699
3	0.6788	0.7145	0.7502	0.7859	0.8216	0.8572	0.8929	0.9286	0.9643
4	0.6416	0.6814	0.7213	0.7611	0.8009	0.8407	0.8805	0.9204	0.9602
5	0.6122	0.6553	0.6984	0.7415	0.7846	0.8277	0.8707	0.9138	0.9569
6	0.5881	0.6338	0.6796	0.7254	0.7711	0.8169	0.8627	0.9085	0.9542
7	0.5676	0.6157	0.6637	0.7117	0.7598	0.8078	0.8559	0.9039	0.9520
8	0.5500	0.6000	0.6500	0.7000	0.7500	0.8000	0.8500	0.9000	0.9500
9	0.5346	0.5863	0.6380	0.6897	0.7414	0.7931	0.8449	0.8966	0.9483
10	0.5208	0.5741	0.6273	0.6806	0.7338	0.7870	0.8403	0.8935	0.9468
11	0.5086	0.5632	0.6178	0.6724	0.7270	0.7816	0.8362	0.8908	0.9454
12	0.4974	0.5533	0.6091	0.6650	0.7208	0.7766	0.8325	0.8883	0.9442
13	0.4873	0.5443	0.6012	0.6582	0.7152	0.7721	0.8291	0.8861	0.9430
14	0.4780	0.5360	0.5940	0.6520	0.7100	0.7680	0.8260	0.8840	0.9420
15	0.4695	0.5284	0.5874	0.6463	0.7053	0.7642	0.8232	0.8821	0.9411
16	0.4615	0.5214	0.5812	0.6410	0.7009	0.7607	0.8205	0.8803	0.9402
17	0.4542	0.5148	0.5755	0.6361	0.6968	0.7574	0.8181	0.8787	0.9394
18	0.4473	0.5087	0.5701	0.6315	0.6929	0.7543	0.8158	0.8772	0.9386
19	0.4408	0.5030	0.5651	0.6272	0.6894	0.7515	0.8136	0.8757	0.9379
20	0.4348	0.4976	0.5604	0.6232	0.6860	0.7488	0.8116	0.8744	0.9372
21	0.4291	0.4925	0.5559	0.6194	0.6828	0.7462	0.8097	0.8731	0.9366
22	0.4237	0.4877	0.5517	0.6158	0.6798	0.7438	0.8079	0.8719	0.9360
23	0.4186	0.4832	0.5478	0.6124	0.6770	0.7416	0.8062	0.8708	0.9354
24	0.4137	0.4788	0.5440	0.6091	0.6743	0.7394	0.8046	0.8697	0.9349
25	0.4091	0.4747	0.5404	0.6061	0.6717	0.7374	0.8030	0.8687	0.9343
26	0.4047	0.4708	0.5370	0.6031	0.6693	0.7354	0.8016	0.8677	0.9339
27	0.4005	0.4671	0.5337	0.6003	0.6669	0.7336	0.8002	0.8668	0.9334
28	0.3965	0.4636	0.5306	0.5977	0.6647	0.7318	0.7988	0.8659	0.9329
29	0.3927	0.4602	0.5276	0.5951	0.6626	0.7301	0.7976	0.8650	0.9325
30	0.3890	0.4569	0.5248	0.5927	0.6606	0.7284	0.7963	0.8642	0.9321
31	0.3855	0.4538	0.5220	0.5903	0.6586	0.7269	0.7952	0.8634	0.9317
32	0.3821	0.4507	0.5194	0.5881	0.6567	0.7254	0.7940	0.8627	0.9313
33	0.3788	0.4478	0.5169	0.5859	0.6549	0.7239	0.7929	0.8620	0.9310
34	0.3757	0.4451	0.5144	0.5838	0.6532	0.7225	0.7919	0.8613	0.9306
35	0.3727	0.4424	0.5121	0.5818	0.6515	0.7212	0.7909	0.8606	0.9303
36	0.3697	0.4398	0.5098	0.5798	0.6499	0.7199	0.7899	0.8599	0.9300
37	0.3669	0.4373	0.5076	0.5780	0.6483	0.7186	0.7890	0.8593	0.9297
38	0.3642	0.4349	0.5055	0.5761	0.6468	0.7174	0.7881	0.8587	0.9294
39	0.3616	0.4325	0.5034	0.5744	0.6453	0.7163	0.7872	0.8581	0.9291
40	0.3590	0.4302	0.5015	0.5727	0.6439	0.7151	0.7863	0.8576	0.9288
41	0.3566	0.4280	0.4995	0.5710	0.6425	0.7140	0.7855	0.8570	0.9285
42	0.3542	0.4259	0.4977	0.5694	0.6412	0.7130	0.7847	0.8565	0.9282
43	0.3518	0.4239	0.4959	0.5679	0.6399	0.7119	0.7839	0.8560	0.9280
44	0.3496	0.4218	0.4941	0.5664	0.6387	0.7109	0.7832	0.8555	0.9277
45	0.3474	0.4199	0.4924	0.5649	0.6374	0.7099	0.7825	0.8550	0.9275
46	0.3453	0.4180	0.4908	0.5635	0.6363	0.7090	0.7818	0.8545	0.9273
47	0.3432	0.4162	0.4891	0.5621	0.6351	0.7081	0.7811	0.8540	0.9270
48	0.3412	0.4144	0.4876	0.5608	0.6340	0.7072	0.7804	0.8536	0.9268
49	0.3392	0.4126	0.4860	0.5595	0.6329	0.7063	0.7797	0.8532	0.9266
50	0.3373	0.4109	0.4846	0.5582	0.6318	0.7055	0.7791	0.8527	0.9264

Table 3: Critical value C_0 for m=2(1)50 and $c = 0.1(0.1)0.9$ at $\alpha = 0.05$

m	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5	c=0.6	c=0.7	c=0.8	c=0.9
1	0.6996	0.7330	0.7663	0.7997	0.8331	0.8665	0.8999	0.9332	0.9666
2	0.6206	0.6627	0.7049	0.7470	0.7892	0.8314	0.8735	0.9157	0.9578
3	0.5711	0.6188	0.6664	0.7141	0.7617	0.8094	0.8570	0.9047	0.9523
4	0.5357	0.5873	0.6389	0.6905	0.7421	0.7936	0.8452	0.8968	0.9484
5	0.5084	0.5630	0.6176	0.6723	0.7269	0.7815	0.8361	0.8908	0.9454
6	0.4864	0.5434	0.6005	0.6576	0.7146	0.7717	0.8288	0.8859	0.9429
7	0.4680	0.5271	0.5862	0.6453	0.7045	0.7636	0.8227	0.8818	0.9409
8	0.4524	0.5132	0.5741	0.6349	0.6958	0.7566	0.8175	0.8783	0.9392
9	0.4389	0.5012	0.5636	0.6259	0.6883	0.7506	0.8130	0.8753	0.9377
10	0.4269	0.4906	0.5543	0.6180	0.6816	0.7453	0.8090	0.8727	0.9363
11	0.4163	0.4812	0.5460	0.6109	0.6757	0.7406	0.8054	0.8703	0.9351
12	0.4068	0.4727	0.5387	0.6046	0.6705	0.7364	0.8023	0.8682	0.9341
13	0.3982	0.4651	0.5320	0.5988	0.6657	0.7325	0.7994	0.8663	0.9331
14	0.3904	0.4581	0.5259	0.5936	0.6613	0.7291	0.7968	0.8645	0.9323
15	0.3832	0.4517	0.5203	0.5888	0.6573	0.7259	0.7944	0.8629	0.9315
16	0.3765	0.4458	0.5151	0.5844	0.6536	0.7229	0.7922	0.8615	0.9307
17	0.3704	0.4404	0.5103	0.5803	0.6502	0.7202	0.7901	0.8601	0.9300
18	0.3647	0.4353	0.5059	0.5765	0.6470	0.7176	0.7882	0.8588	0.9294
19	0.3594	0.4305	0.5017	0.5729	0.6441	0.7153	0.7865	0.8576	0.9288
20	0.3544	0.4261	0.4978	0.5696	0.6413	0.7130	0.7848	0.8565	0.9283
21	0.3497	0.4219	0.4942	0.5664	0.6387	0.7110	0.7832	0.8555	0.9277
22	0.3452	0.4180	0.4907	0.5635	0.6362	0.7090	0.7817	0.8545	0.9272
23	0.3411	0.4143	0.4875	0.5607	0.6339	0.7071	0.7804	0.8536	0.9268
24	0.3371	0.4108	0.4844	0.5581	0.6317	0.7054	0.7790	0.8527	0.9263
25	0.3334	0.4074	0.4815	0.5556	0.6297	0.7037	0.7778	0.8519	0.9259
26	0.3298	0.4043	0.4788	0.5532	0.6277	0.7021	0.7766	0.8511	0.9255
27	0.3264	0.4013	0.4761	0.5510	0.6258	0.7006	0.7755	0.8503	0.9252
28	0.3232	0.3984	0.4736	0.5488	0.6240	0.6992	0.7744	0.8496	0.9248
29	0.3201	0.3957	0.4712	0.5467	0.6223	0.6978	0.7734	0.8489	0.9245
30	0.3172	0.3930	0.4689	0.5448	0.6206	0.6965	0.7724	0.8483	0.9241
31	0.3143	0.3905	0.4667	0.5429	0.6191	0.6953	0.7714	0.8476	0.9238
32	0.3116	0.3881	0.4646	0.5411	0.6176	0.6941	0.7705	0.8470	0.9235
33	0.3090	0.3858	0.4626	0.5393	0.6161	0.6929	0.7697	0.8464	0.9232
34	0.3065	0.3836	0.4606	0.5377	0.6147	0.6918	0.7688	0.8459	0.9229
35	0.3041	0.3814	0.4588	0.5361	0.6134	0.6907	0.7680	0.8454	0.9227
36	0.3018	0.3794	0.4569	0.5345	0.6121	0.6897	0.7673	0.8448	0.9224
37	0.2995	0.3774	0.4552	0.5330	0.6109	0.6887	0.7665	0.8443	0.9222
38	0.2974	0.3755	0.4535	0.5316	0.6097	0.6877	0.7658	0.8439	0.9219
39	0.2953	0.3736	0.4519	0.5302	0.6085	0.6868	0.7651	0.8434	0.9217
40	0.2933	0.3718	0.4503	0.5289	0.6074	0.6859	0.7644	0.8430	0.9215
41	0.2913	0.3701	0.4488	0.5276	0.6063	0.6850	0.7638	0.8425	0.9213
42	0.2894	0.3684	0.4473	0.5263	0.6052	0.6842	0.7631	0.8421	0.9210
43	0.2876	0.3668	0.4459	0.5251	0.6042	0.6834	0.7625	0.8417	0.9208
44	0.2858	0.3652	0.4445	0.5239	0.6032	0.6826	0.7619	0.8413	0.9206
45	0.2841	0.3636	0.4432	0.5227	0.6023	0.6818	0.7614	0.8409	0.9205
46	0.2824	0.3622	0.4419	0.5216	0.6014	0.6811	0.7608	0.8405	0.9203
47	0.2808	0.3607	0.4406	0.5205	0.6004	0.6804	0.7603	0.8402	0.9201
48	0.2792	0.3593	0.4394	0.5195	0.5996	0.6797	0.7597	0.8398	0.9199
49	0.2777	0.3579	0.4382	0.5185	0.5987	0.6790	0.7592	0.8395	0.9197
50	0.2762	0.3566	0.4370	0.5175	0.5979	0.6783	0.7587	0.8392	0.9196

Since $CHIINV(1-\alpha, 2m)$ represents the lower $(1-\alpha)$ percentile of χ^2_{2m}, the level $(1-\alpha)$ one-sided confidence interval for C_L can be derived as follows:

$$\begin{aligned} P(2\lambda W \leq CHIINV(1-\alpha, 2m)) = 1-\alpha, \quad & where\ C_L = 1-\lambda L\ and\ \hat{C}_L = 1-\frac{mL}{W} \\ & \Rightarrow P\left(1-\lambda L \geq 1-\frac{\hat{\lambda} L\ CHIINV(1-\alpha, 2m)}{2m}\right) = 1-\alpha, \\ & \Rightarrow P\left(C_L \geq 1-\frac{(1-\hat{C}_L)CHIINV(1-\alpha, 2m)}{2m}\right) = 1-\alpha, \end{aligned} \tag{21}$$

where $\hat{\lambda} = \frac{m}{W}$. From the equation (21),

$$C_L \geq 1-\frac{(1-\hat{C}_L)CHIINV(1-\alpha, 2m)}{2m} \tag{22}$$

is the level $(1-\alpha)$ one-sided confidence interval for C_L.

Thus, the lower confidence bound for C_L at the level $(1-\alpha)$ can be written as

$$LB = 1-\frac{(1-\hat{C}_L)CHIINV(1-\alpha, 2m)}{2m} \tag{23}$$

where $\hat{C_L}$, α and m denote the MLE of C_L, the specified significance level and the number of observed failures before termination, respectively.

If the performance index value $c \notin [LB, \infty)$, we can conclude that the lifetime performance index of the product meets the required level.

5 The Monte Carlo simulation algorithm of the power function

The power of this statistical test is the probability of correctly rejecting a false null hypothesis. the null hypothesis H_0 and the alternative hypothesis H_1 are given as:

$$H_0 : C_L \leq c \qquad VS \qquad H_1 : C_L > c. \tag{24}$$

We refer the algorithm proposed by Lee (2009) and Balakrishnan Algorithm on progressive censoring data to find the power of the statistical test as follows:
We get a size α test with the rejection region $\{\hat{C}_L|\hat{C}_L > 1-\frac{2m(1-c)}{CHIINV(1-\alpha,2m)}\}$ for the number of observed failures before termination m and the sample size n $(m \leqslant n)$. The power $P(c_1)$ of the test at $C_L = c_1(> c)$ is

$$
\begin{aligned}
P(c_1) &= P\left(\hat{C}_L = 1 > 1 - \frac{2m(1-c)}{CHIINV(1-\alpha,2m)} \,\middle|\, C_L = c_1\right) \\
&= P\left(1 - \frac{mL}{W} > 1 - \frac{2m(1-c)}{CHIINV(1-\alpha,2m)} \,\middle|\, \lambda = \frac{1-c_1}{L}\right) \\
&= P\left(2\lambda W > \frac{\lambda\ L\ CHIINV(1-\alpha,2m)}{1-c} \,\middle|\, \lambda = \frac{1-c_1}{L}\right) \\
&= P\left(2\lambda W > \frac{(1-c)CHIINV(1-\alpha,2m)}{1-c}\right)
\end{aligned}
$$

where $2\lambda W \sim \chi^2_{(2m)}$.

The Monte Carlo simulation algorithm of the power $P(c_1)$ is given in the following steps:

Step 1:

(a) Given c, c_1, L, α, m, n, and R $=(R_1, R_2, ..., R_m)$, where $c < c_1 < 1$ and m $\leqslant n$

(b) The value of λ is calculated by the equation $C_L = 1 - \lambda L = c_1, C_L < 1$.

(c) The generation of data $U_1, U_2, ..., U_m$ is by uniform distribution U(0,1).

(d) By the transformation of $Z_i = -ln(1 - U_i), i = 1, ..., m, (Z_1.Z_2, ..., Z_m)$ is a random sample from the exponential distribution with p.d.f as (1).

(e) Set
$X'_{i,n} = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + ... + \frac{Z_i}{n-R_1-R_2-...-R_{i-1}-i+1}$, for $i = 1, ..., m$.

$(X'_{1,n}, X'_{2.n}, ..., X'_{m,n})$ is the progressively type II right censored sample from a one-parameter exponential distribution.

(f) Next, set
$X_{i,n} = F^{-1}[1 - exp(-X'_{i,n})]$, for $i = 1, 2, ..., m$, where $F^{-1}(.)$ is the inverse cumulative distributions function of the two-parameter Rayleigh distribution. Then $X_{1,n}, X_{2,n}, ..., X_{m,n}$ is the required progressively Type II right censored sample from two parameter Rayleigh distribution.

(g) Now apply the transformation $Y_{i,n} = (X_{i,n} - \mu)^2$

(h) The value of $\hat{C}_L$ is calculated by $\hat{C}_L = 1 - \frac{mL}{\sum_{i=1}^{m}(1+R_i)Y_{i,n}}$

(i) If $\hat{C}_L > C_0$ then Count= 1, else Count= 0, where $C_0 = 1 - \frac{2m(1-c)}{CHIINV(1-\alpha,2m)}$.

Step 2:

(a) The step 1 is repeated 1000 times.

(b) The estimation of power $P(c_1) is \hat{P}(c_1) = \frac{Total\ \ Count}{1000}$

Step 3:

(a) The step 2 is repeated 100 times, then 100 estimators of the power $P(c_1)$ can be obtained as follows:

$$\hat{P}_1(c_1), \hat{P}_1(c_1), ..., \hat{P}_{100}(c_1).$$

(b) The mean $\overline{\hat{P}(c_1)} of \hat{P}_1(c_1), \hat{P}_2(c_2), ..., \hat{P}_{100}(c_1)$, that is

$$\overline{\hat{P}(c_1)} = \frac{\sum_{i=1}^{100} \hat{P}_i(c_1)}{100}.$$

(c) The sample mean square error(SMSE) of $\hat{P}_1(c_1), \hat{P}_2(c_2), ..., \hat{P}_{100}(c_1)$ is

$$SMSE = \frac{\sum_{i=1}^{100} (\hat{P}_i(c_1) - P(c_1))^2}{100}, \text{where} P(c_1) \text{can be calculated by (16)}$$

Table 4: The values of $P(c_1)$, $\overline{\hat{P}(c_1)}$ and $SMSE$ for the different n and $r(\alpha = 0.05)$

c_1	$P(c_1)$	$\overline{\hat{P}(c_1)}$	$SMSE$	$P(c_1)$	$\overline{\hat{P}(c_1)}$	$SMSE$	$P(c_1)$	$\overline{\hat{P}(c_1)}$	$SMSE$
	n=10, m=5 R=(1,1,0,1,2)			n=15, m=5 R=(2,1,1,2,4)			n=15, m=10 R=(0,1,0,1,0,2,0,0,0,1)		
0.1	0.05000	0.05093	0.00005	0.05000	0.05093	0.00005	0.05000	0.05084	0.00005
0.2	0.09208	0.09434	0.00011	0.09208	0.09434	0.00011	0.11130	0.11178	0.00011
0.3	0.16237	0.16510	0.00015	0.16237	0.16510	0.00015	0.22410	0.22559	0.00023
0.4	0.27159	0.27421	0.00023	0.27159	0.27421	0.00023	0.40066	0.40299	0.00026
0.5	0.42565	0.42632	0.00025	0.42566	0.42632	0.00026	0.62357	0.62510	0.00027
0.6	0.61551	0.61713	0.00027	0.61551	0.61713	0.00027	0.83251	0.83237	0.00013
0.7	0.80659	0.80723	0.00013	0.80659	0.80723	0.00013	0.95882	0.95932	0.00003
0.8	0.94422	0.94477	0.00003	0.94422	0.94477	0.00003	0.99675	0.99677	0.00000
0.9	0.99607	0.99597	0.00000	0.99607	0.99597	0.00000	0.99999	0.99999	0.00000

Table 5: The values of $P(c_1)$, $\overline{\hat{P}(c_1)}$ and $SMSE$ for the different n and $r(\alpha = 0.05)$

c_1	$P(c_1)$	$\hat{P}(c_1)$	$SMSE$	$P(c_1)$	$\hat{P}(c_1)$	$SMSE$	$P(c_1)$	$\hat{P}(c_1)$	$SMSE$
	n=20, m=5 R=(0,1,3,1,10)			n=20, m=10 R=(0,1,0,2,1,3,0,2,0,1)			n=20, m=15 R=(0,1,0,1,1,0,0,0,0,1,0,0,0,1,0)		
0.1	0.05000	0.05093	0.00005	0.05000	0.05084	0.00005	0.05000	0.04981	0.00004
0.2	0.092087	0.09434	0.00011	0.11130	0.11178	0.00011	0.12777	0.12758	0.00011
0.3	0.162375	0.16510	0.00015	0.22410	0.22559	0.00023	0.27901	0.28072	0.00020
0.4	0.27159	0.27421	0.00023	0.40065	0.40299	0.00026	0.50805	0.50970	0.00023
0.5	0.42566	0.42632	0.00026	0.62357	0.62510	0.00027	0.75744	0.76020	0.00015
0.6	0.61551	0.61713	0.00027	0.83251	0.83237	0.00013	0.92997	0.93067	0.00005
0.7	0.80659	0.80723	0.00013	0.95882	0.95932	0.00003	0.99185	0.99240	0.00000
0.8	0.94422	0.94477	0.00003	0.99675	0.99677	0.00000	0.99983	0.99979	0.00000
0.9	0.99607	0.99597	0.00000	0.99999	0.99999	0.00000	0.10000	0.10000	0.00000

Table 6: The values of $P(c_1)$, $\overline{\hat{P}(c_1)}$ and $SMSE$ for the different n and $r(\alpha = 0.01)$

c_1	$P(c_1)$	$\overline{\hat{P}(c_1)}$	$SMSE$	$P(c_1)$	$\overline{\hat{P}(c_1)}$	$SMSE$	$P(c_1)$	$\overline{\hat{P}(c_1)}$	$SMSE$
	n=10, m=5 R=(1,1,0,1,2)			n=15, m=5 R=(2,1,1,2,4)			n=15, m=10 R=(0,1,0,1,0,2,0,0,0,1)		
0.1	0.01000	0.01011	0.00001	0.01000	0.01011	0.00001	0.01000	0.01034	0.00001
0.2	0.02382	0.02356	0.00001	0.02382	0.02356	0.00001	0.03054	0.03146	0.00003
0.3	0.05409	0.05537	0.00006	0.05410	0.05537	0.00006	0.08355	0.08408	0.00009
0.4	0.11575	0.11792	0.00012	0.11575	0.11792	0.00012	0.19975	0.20169	0.00022
0.5	0.22966	0.23203	0.00020	0.22966	0.23203	0.00020	0.40481	0.40711	0.00025
0.6	0.41329	0.41401	0.00023	0.41329	0.41401	0.00023	0.67261	0.67371	0.00020
0.7	0.65457	0.65582	0.00025	0.65457	0.65582	0.00025	0.89694	0.89695	0.00007
0.8	0.88040	0.88218	0.00010	0.88040	0.88218	0.00010	0.98930	0.98974	0.00001
0.9	0.98968	0.98971	0.00001	0.98968	0.98971	0.00001	0.99993	0.99996	0.00000

Table 7: The values of $P(c_1)$, $\overline{\hat{P}(c_1)}$ and $SMSE$ for the different n and $r(\alpha = 0.01)$

c_1	$P(c_1)$	$\hat{P}(c_1)$	$SMSE$	$P(c_1)$	$\hat{P}(c_1)$	$SMSE$	$P(c_1)$	$\hat{P}(c_1)$	$SMSE$
	n=20, m=5 R=(0,1,3,1,10)			n=20, m=10 R=(0,1,0,2,1,3,0,2,0,1)			n=20, m=15 R=(0,1,0,1,1,0,0,0,0,1,0,0,0,1,0		
0.1	0.01000	0.01011	0.00001	0.01000	0.01034	0.00001	0.01000	0.00991	0.00000
0.2	0.02382	0.02356	0.00001	0.03054	0.03146	0.00003	0.03668	0.03689	0.00003
0.3	0.05409	0.05537	0.00006	0.08355	0.08408	0.00009	0.11320	0.11257	0.00011
0.4	0.11575	0.11792	0.00013	0.19975	0.20169	0.00022	0.28372	0.28543	0.00020
0.5	0.22966	0.23203	0.00021	0.40481	0.40711	0.00025	0.55595	0.55762	0.00022
0.6	0.41329	0.41401	0.00024	0.6726	0.67371	0.00020	0.83060	0.83365	0.00015
0.7	0.65457	0.65582	0.00025	0.89693	0.89695	0.00007	0.97300	0.97292	0.00002
0.8	0.88040	0.88218	0.00010	0.98930	0.98974	0.00001	0.99921	0.99920	0.00000
0.9	0.98968	0.98971	0.00001	0.99993	0.99997	0.00000	0.10000	0.10000	0.00000

From Tables 4 to 7 show the result of simulations that compared $\hat{P}(c_1)$ with $P(c_1)$ for given values of $(n, m) = (10, 5), (15, 5), (15, 10), (20, 5), (20, 10), (20, 15), c = 0.1, c_1 = 0.2(0.1)0.9$ and $L = 1$ at $\alpha = 0.05$ and $\alpha = 0.01$.

Based on the above results, as the observed number before termination m increases the simulation power $\hat{P}(c_1)$ and the power $P(c_1)$ increases for fixed c_1 except $c_1 = 0.1$. When c_1 increases, the simulation power $\hat{P}(c_1)$ and the power $P(c_1)$ increases for fixed m. Both the simulation power $\hat{P}(c_1)$ and the power $P(c_1)$ are very close for each case. Scope of $SMSE$ is between 0.00000 and 0.00027, so $SMSE$ are enough small.

6 Concluding Remarks

Under the assumption of two parameter Raileigh distribution, applying the data transformation, we developed a MLE of C_L with the progressively type II right censored sample. Then we utilized the MLE of C_L to construct a hypothesis testing procedure. Based on the results of critical values and power simulation, the proposed testing procedure can be effectively utilized to assess the product process capability.

7 Acknowledgement

The authors would like to thank the editors and the referees for their helpful suggestions and comments which improved this paper.

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