Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
`http://siba-ese.unisalento.it/index.php/ejasa/index`
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v12n1p190

Estimating parameters of Morgenstern type bivariate distribution using bivariate ranked set sampling

By Al Kadiri, Migdadi

Published: 26 April 2019

Electronic Journal of Applied Statistical Analysis
Vol. 12, Issue 01, April 2019, 190-208
DOI: 10.1285/i20705948v12n1p190

Estimating parameters of Morgenstern type bivariate distribution using bivariate ranked set sampling

Al Kadiri M. A.*[a] and Migdadi M. K.[b]

[a] *Department of Statistics, Yarmouk University, Irbid-Jordan*
[b] *Common First Year Department, King Saud University, Riyadh-Saudi Arabia*

Published: 26 April 2019

This paper improves estimating parameters of Morgenstern type bivariate distribution by developing bivariate ranked set sampling procedure as an alternative method to simple random sampling. This proposed procedure gives an opportunity to estimate all distribution's parameters simultaneously which is not investigated in previous studies, yet. Simulation studies are conducted to investigate properties of the new estimators and compare them with some other existed estimators.

keywords: Morgenstern type bivariate distribution, Bivariate Ranked Set Sampling, concomitant ordered statistics, Average relative estimate, Relative efficiency.

1 Introduction

Assume X and Y are bivariate random variables with moderate association. A suitable joint probability density function "pdf" that can accommodate this association was suggested by Morgenstern (1956) as:

$$f_{X,Y}(x,y,\theta,\beta) = f_X(x,\theta) f_Y(y,\beta)\Big[1+\alpha\Big(1-2F_X(x,\theta)\Big)\Big(1-2F_Y(y,\beta)\Big)\Big] \tag{1}$$

where $f_X(x,\theta)$ and $f_Y(y,\beta)$ are the pdfs for X and Y respectively, $F_X(x,\theta)$ and $F_Y(y,\beta)$ are their correspondence Distribution Functions "DF"s, and θ, β and α are model parameters. The main parameter for this model is the association parameter α which is

*Corresponding author: alkadiri-m@yu.edu.jo

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

proportionally related to the correlation coefficient between the two variables of interest and its range $-1 \leq \alpha \leq 1$. This joint pdf in (1) known in the literature by Morgenstern Type Bivariate Distribution ""MTBD"".

Specific examples on this bivariate densities of correlated random variables are Morgenstern Type Bivariate Uniform Distribution "MTBUD" and Morgenstern Type Bivariate Exponential Distribution "MTBED" . Their pdfs respectively are:

$$f_{X,Y}(x, y, \theta, \beta) = \frac{1}{\theta}\frac{1}{\beta}\Big[1 + \alpha\Big(1 - \frac{2x}{\theta}\Big)\Big(1 - \frac{2y}{\beta}\Big)\Big], 0 < x < \theta; 0 < y < \beta \tag{2}$$

and

$$f_{X,Y}(x, y, \theta, \beta) = \frac{1}{\theta}\frac{1}{\beta}e^{(\frac{-x}{\theta}+\frac{-y}{\beta})}\Big[1 + \alpha\Big(1 - 2e^{\frac{-x}{\theta}}\Big)\Big(1 - 2e^{\frac{-2y}{\beta}}\Big)\Big], 0 < x; 0 < y. \tag{3}$$

Researchers paid attention on estimating parameters of MTBD for decades. For example, Scaria and Nair (1999) depended on the concomitant of ordered statistics to estimate β and α. Concomitant of ordered statistics means ordering values of a random variable made according to corresponding values of another random variable which are ranked perfectly (i.e. ordered statistics).

Chacko and Thomas (2008) and Tahmasebi and Jafari (2014) estimated the same parameters when the underlying distribution is MTBED using a sampling technique called Ranked Set Sampling ""RSS"". Also, they proposed other modifications on this sampling approach to gain estimation. Singh and Mehta (2016) improved estimating parameters when the underlying distribution is MTBUD using concomitant of ordered statistics.

Beside establishing theory for general concomitant of generalized order statistics from generalized MTBD family, Domma and Giardano (2016) derived moments and recurrence relations between moments of concomitant of ordered statistics to estimate MTBD's parameters. Genest et al. (1995), Abo-Eleneen and Nagaraja (2002) and Stefanescu and Turnbull (2009) investigated estimating the association parameter in MTBD using maximum likelihood approach or one of its modifications while Al Kadiri and Migdadi (2018) proposed the same approach to estimate the association parameter using a modified maximum likelihood approach.

Ranked sampling, such as RSS and BVRSS, are used as alternative sampling methods to Simple Random Sampling "SRS". Mainly, ranked sampling depends on ordering a small set of sample units visually or by a cheap method then measure only a subset of the ordered units.These sampling techniques improve properties of the produced estimators as well as can reduce sampling costs, Wolfe (2012). Modifications on RSS can potentially facilitate method of selection as well. Al-Saleh and Al Kadiri (2000) and Al-Saleh and Al-omari (2002), for example, made spots on similar issues.

In this paper, we develop BVRSS procedure then we examine the usefulness of these modifications to improve estimating all MTBD parameters. We divide this sampling method into two phases. The first phase produces copies of concomitant of ordered statistics while the second phase produces the regular BVRSS units. As known, BVRSS depends on sample units of the second phase to achieve estimation while in this paper, we apply both phases to gain parameter estimation. Additionally, we build theoretical infrastructure needed to investigate properties of new estimators.

BRSS was introduced by Al-Saleh and Zheng (2002) as a bivariate version of the usual RSS to be used when we deal with two characteristics jointly.

In the following, we describe this procedure.

Suppose (X_i, Y_i) is a bivariate random variables with a joint probability density function $f_{X,Y}(x, y)$. To generate a BVRSS sample, we need to do the following steps:

Step 1: For a set of size m, select a "SRS" of size m^4 from the population and divide it randomly into m^2 pools, each pool has $m \times m$ units. The elements in the first pool are denoted by $(X_{ij}^{(1)}, Y_{ij}^{(1)}), i = 1, 2, ..., m; j = 1, 2, ..., m,$ where $X_{ij}^{(1)}$ and $X_{ij}^{(1)}$ are the j^{th} element of the i^{th} row in the first pool for the first characteristic and the second characteristic, respectively.

Step 2: For each row in the first pool, identify the minimum value based on first characteristic by judgment. In symbols $(X_{i(j)}^{(1)}, Y_{i[j]}^{(1)}), i = 1, 2, ..., m; j = 1, 2, ..., m.$ This step produces the first phase sampling in our paper such that $Y_{i[j]}^{(1)}$ is concomitant of the ordered statistic $X_{i(j)}^{(1)}$. The round brackets on the lower subscripts of the letters means that the ranking is perfect (i.e. ordered statistics) while square brackets means ordering corresponds to the perceived orders that match the other variable (i.e. concomitant ordered statistics).

Step 3: For the m minima obtained in Step 2, choose the pair that corresponds to the minimum value of the second characteristic, identified by judgement, for actual quantification. This pair labelled by (1,1) in the second phase BVRSS.

Step 4: Repeat Steps 2 and 3 for the second pool, but in Step 3, the pair that corresponds to the second minimum value with respect to the second characteristic is chosen for actual quantification. This pair labelled by (1,2).

Step 5: The process continues until the label (m, m) is selected from the m^{th} (last) pool. In symbols $(X_{[i](j)}^{(k)}, Y_{(i)[j]}^{(k)}), i = 1, 2, ..., m; j = 1, 2, ..., m$ and $k = (j-1)m + i$.

Step 6: For sampling comparison purposes, larger sample size is possibly required. So we can repeat the above steps r times to obtain a sample of size $n = rm^2$. Here, n represents the SRS sample size.

The above method produces an independent but not identically distributed BVRSS sample of size m^2, which its units are denoted by $(X_{[i](j)}^{(k)}, Y_{(i)[j]}^{(k)})$. Note that m^4 units are selected to measure m^2 for actual quantification. Considerably, the m^4 sample units contribute the information to m^2 quantified units.

This paper is established as follows. Section (2) builds a theoretical infrastructure needed to achieve parameter estimation in this paper. Then, Section (3) improves estimating parameters of MTBD in general and for a few specific examples while Section (4) presents simulation studies to illustrate properties of estimators. A brief discussion and conclusions come in Section (5).

2 Basic theory setups

Let (X_i, Y_i) be a sequence of bivariate SRS sample with joint MTBD pdf in (1) where $i = 1, 2, ..., n$. Consider μ_X and μ_Y are population means and $var_X(x)$ and $var_Y(y)$ are population variances.

Also, let $(X^{(k)}_{i(j)}, Y^{(k)}_{i[j]})$ be the produced first stage BVRSS sample from the k^{th} pool where $i = 1, 2, ..., m$ and $j = 1, 2, ..., m$. Consider their correspondence means $E_{X_{(j:m)}}(x) = \mu_{X_{(j:m)}}$, $E_{Y_{[j:m]}}(x) = \mu_{Y_{[j:m]}}$ and correspondence variances $var_{X_{(j:m)}}(x)$ and $var_{Y_{[j:m]}}(x)$. The produced samples from this stage are copies of concomitant of ordered statistics where each pool contains m copies.

Scaria and Nair (1999) defined pdf of the first stage concomitant variable $Y_{[j:m]}$ for MTBD as follows:

$$f_Y(y, \beta) = f_Y(y, \beta)\Big[1 + \alpha\ A_{j,m}\Big(1 - 2F_Y(y, \beta)\Big)\Big] \text{where } A_{j,m} = \frac{m - 2j + 1}{m + 1}.$$

Note that the pdf of $X_{(j:m)}$ is the usual pdf of the j^{th} ordered statistic (see for example Ehsanullah et al. (2013); page 19).

Similarly, assume $(X^{(k)}_{[i:m](j)}, Y^{(k)}_{(i:m)[j]})$ be the produced second stage BVRSS sample from the k^{th} pool such that $i = 1, 2, ..., m$ and $j = 1, 2, ..., m$. Consider means of these random variables are $E_{X_{[i:m](j)}}(x) = \mu_{X_{[i:m](j)}}$, $E_{Y_{(i:m)[j]}}(x) = \mu_{Y_{(i:m)[j]}}$ and their variances are $var_{X_{[i:m](j)}}(x)$ and $var_{Y_{(i:m)[j]}}(x)$ respectively.

Note that the pdf of $X^{(k)}_{[i:m](j)}$ is the density of the i^{th} concomitant random variable when it is selected from a set of ordered statistics with rank j (i.e. not SRS sample). Also, the pdf of $Y^{(k)}_{(i:m)[j]}$ is the density of the i^{th} ordered statistics when this random variable is selected from a set of concomitant variables with rank j. Marginal and joint pdfs of these two random variables are settled in the following result.

Result 2.1. *The marginal pdf of the random variable $X^{(k)}_{[i:m](j)}$ can be expressed as*

$$f_{X_{[i:m](j)}}(x, \theta) = f_{X_{(j)}}(x, \theta)[1 + \alpha\ A_{i,m}(1 - 2\ F_{X_{(j)}}(x, \theta))] \tag{4}$$

where $f_{X_{(j)}}(x, \theta)$ and $F_{X_{(j)}}(x, \theta)$ are the pdf and DF of the ordered statistic $X_{(j)}$ respectively. Furthermore, the marginal pdf of the random variable $Y^{(k)}_{(i:m)[j]}$ can be written as

$$f_{Y_{(i:m)[j]}}(y, \beta) = C_{m,i}[F_{Y_{[j:m]}}(y, \beta)]^{i-1}[1 - F_{Y_{[j:m]}}(y, \beta)]^{m-i} f_{Y_{[j:m]}}(y, \beta) \tag{5}$$

where $F_{Y_{[j:m]}}(y, \beta)$ is the DF of the j^{th} concomitant variable and the constant $C_{a,b} = \frac{a!}{(b-1)!(a-b)!}$. Also, the joint density can be written as

$$\begin{aligned} f_{X_{[i:m](j)}, Y_{(i:m)[j]}}(x, y; \theta, \beta, \alpha) &= C_{m,i} f_{X_{(j:m)}}(x, \theta)[F_{Y_{[j:m]}}(y, \beta)]^{i-1}[1 - F_{Y_{[j:m]}}(y, \beta)]^{m-i} \\ & \quad f_{Y|X}(y|x) \end{aligned} \tag{6}$$

where the conditional density in (6) is:
$f_{Y|X}(y|x) = f_Y(y; \beta)[1 + \alpha(1 - 2F_X(x, \theta))(1 - 2F_Y(y; \beta))]$.

Definition 2.1. 1) Under general distribution assumption, Mclntyre (1952) estimated the population mean μ_X by the unbiased estimator $\hat{\mu}_X^{(1)} = \frac{1}{m}\sum_{j=1}^{m} X_{(j:m)}^{(k)}$. While Scaria and Nair (1999) estimated the population mean μ_Y for specific MTBD by the unbiased estimator $\hat{\mu}_Y^{(1)} = \frac{1}{m}\sum_{j=1}^{m} Y_{[j:m]}^{(k)}$.

2) Al-Saleh and Zheng (2002) proposed the following unbiased estimators for population means: $\hat{\mu}_X^{(2)} = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} X_{[i:m](j)}^{(k)}$and $\hat{\mu}_Y^{(2)} = \frac{1}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{m} Y_{(i:m)[j]}^{(k)}$.

The previous definition, in its two parts, comes from basic identities of pdfs which are summarized in the following remark.

Remark: 1) (Takahasi and Wakimoto (1968)) $f_X(x) = \frac{1}{m}\sum_{j=1}^{m} f_{X_{(j:m)}}(x)$ and similarly $f_Y(y) = \frac{1}{m}\sum_{j=1}^{m} f_{Y_{[j:m]}}(Y)$.

2)(Scaria and Nair (1999)) $f_Y(y) = \frac{1}{2}(f_{Y_{[j:m]}}(y)+f_{Y_{[m-j+1:m]}}(y))$ for MTBD specifically.

3)(Al-Saleh and Zheng (2002)) $f_{X,Y}(x,y) = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} f_{X_{[i:m](j)},Y_{(i:m)[j]}}(x,y)$ and therefore, $f_X(x) = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} f_{X_{[i:m](j)}}(x)$ and $f_Y(y) = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} f_{Y_{(i:m)[j]}}(y)$.

Lemma 2.1. *According to Domma and Giardano (2016), we can state that the concomitant variable $Y_{[j:m]}$, from any arbitrary pool, which originally comes from MTBD has the following mean and variance:*
1) $E_{Y_{[j:m]}}[y] = \mu_{Y_{[j:m]}} = \mu_Y + \alpha A_{j,m} E_Y[y(1-2F_Y(y,\beta))]$.
2)$var_{Y_{[j:m]}}(y) =$
$var_Y(y)+\alpha\ A_{j,m}E_Y[y^2(1-2F_Y(y,\beta))]-2\alpha A_{j,m}\mu_Y E_Y[y(1-2F_Y(y,\beta))]-\alpha^2 A_{j,m}^2 E_Y^2[y(1-2F_Y(y))]$.

According to this lemma, Domma and Giardano (2016) suggested the following expression to represent the association parameter α

$$\hat{\alpha} = \frac{Y_{[m-s+1:m]} - Y_{[s:m]}}{2A_{s,m}E_Y[y(1-2F_Y(y,\beta))]} \tag{7}$$

at specific s such that $s = \begin{cases} 1,2,....,\ \left[\frac{m+1}{2}\right]-1 & \text{if } m \text{ is odd} \\ 1,2,....,\ \left[\frac{m+1}{2}\right] & \text{if } m \text{ is even} \end{cases}$. Here the operation $[a]$ means the greatest integer less than or equal a.

Theorem 2.1. *Analogous to Lemma (2.1), we can state the following mean and variance for the concomitant variables of the second stage BVRSS:*
1) The mean $E_{X_{[i:m](j)}}(x) = \mu_{X_{(j:m)}} + \alpha\ A_{i,m}\left(\mu_{X_{(j:m)}} - 2C_{m,j}\sum_{k=j}^{m}\binom{m}{k}\frac{1}{C_{2m,k+j}}\mu_{X_{(k+j:2m)}}\right)$
where $\mu_{X_{(k+j:2m)}} = E_{X_{(k+j:2m)}}(x)$.

2) The variance

$$var_{X_{[i:m](j)}}(x)$$
$$= var_{X_{(j:m)}}(x) + \alpha A_{i,m}\Bigg(E_{X_{(j:m)}}[x^2(1-2F_{X_{(j:m)}}(x))] - 2E_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))] \times$$
$$\left(\mu_{X_{(j:m)}} + \alpha\ A_{i,m}E_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))]\right)\Bigg)$$

where $F_{X_{(j:m)}}(x)$ is the DF of the ordered statistics $X_{(j:m)}$.

Proof: 1) We can define expectation for random variable $X_{[i:m](j)}$ as
$E[X_{[i:m](j)}] = E_{X_{(j:m)}}(x) + \alpha A_{i,m} E_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))]$.

By re-expressing the last term in the above equation as
$E_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))] = \mu_{X_{(j:m)}} - 2E_{X_{(j:m)}}[xF_{X_{(j:m)}}(x)]$
and by using the result by David and Nagaraja (2003), Chapter (2), which is:
$F_{X_{(j)}}(x) = \sum_{k=j}^{m} \binom{m}{k} F_X(x)^k (1-F_X(x))^{m-k}$
we can conclude that,
$E_{X_{(j:m)}}\left[x\left(1-2F_{X_{(j:m)}}(x)\right)\right]$
$= \mu_{X_{(j:m)}} - 2\,C_{m,j}\sum_{k=j}^{m}\binom{m}{k}\int xf_X(x)F_X(x)^{k+j-1}[1-F_X(x)]^{2m-k-j}dx$
$= \mu_{X_{(j:m)}} - 2\,C_{m,j}\sum_{k=j}^{m}\binom{m}{k}\frac{1}{C_{2m,k+j}}\mu_{X_{(k+j:2m)}}$.
Thus, part 1) of this theorem is straightforward.

2) Proof the second part of this theorem is similar to part (2) of Lemma (2.1) ■

Lemma 2.2. *Variances of the unbiased estimators $\hat{\mu}_X^{(2)}$ and $\hat{\mu}_Y^{(2)}$, can be expressed under MTBD as:*
1) $var(\hat{\mu}_X^{(2)}) = \frac{var_X(x)}{m^2} - \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m}(\mu_{X_{[i:m](j)}} - \mu_X)^2$.
2) $var(\hat{\mu}_Y^{(2)}) = \frac{var_Y(y)}{m^2} - \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m}(\mu_{Y()i:m)[j]} - \mu_Y)^2$.

Proof two parts of the above lemma for general distribution assumption can be found in Al-Saleh and Zheng (2002) however, we need variance formulas to include the association parameter of MTBD. So, we develop the following result.

Result 2.2. *Variances of the unbiased estimators $\hat{\mu}_X^{(2)}$ and $\hat{\mu}_Y^{(2)}$ can be re-expressed under MTBD as*

$$1)var(\hat{\mu}_X^{(2)}) = \frac{1}{m^4}\sum_{j=1}^{m} var_{X_{(j:m)}}(x) - \alpha^2\frac{(m-1)}{3m^3(m+1)}\sum_{j=1}^{m} E^2_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))]$$

$$2)var(\hat{\mu}_Y^{(2)}) = \frac{1}{m^4}\sum_{j=1}^{m} var_{Y_{[j:m]}}(y) - \alpha^2\frac{(m-1)}{3m^3(m+1)}\sum_{j=1}^{m} E^2_{Y_{[j:m]}}[y(1-2F_{y_{[j:m]}}(y))].$$

Proof: We prove the first part of the above result while proof of the second part can be done similarly. Since all BVRSS units are independent, we can write:
$var(\hat{\mu}_X^{(2)}) = \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m} var_{X_{[i:m](j)}}(X)$. Applying Theorem 2.1 part 2) we get:

$$
\begin{aligned}
var(\hat{\mu}_X^{(2)}) &= \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m} var_{X_{(j:m)}}(x) + \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m} \alpha A_{i,m} E_{X_{(j:m)}}[x^2(1-2F_{X_{(j:m)}}(x))] \\
&- 2\alpha\frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m} A_{i,m} E_{X_{(j:m)}}[x^2(1-2F_{X_{(j:m)}}(x))]\mu_{X_{(j:m)}} \\
&- \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m} \alpha^2 A_{i,m}^2 E_{X_{(j:m)}}^2[x(1-2F_{X_{(j:m)}}(x))].
\end{aligned}
$$

Since $\sum_{i=1}^{m} A_{i,m} = 0$ and $\sum_{i=1}^{m} A_{i,m}^2 = \frac{m(m-1)}{3(m+1)}$, we get
$var(\hat{\mu}_X^{(2)}) = \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m} var_{X_{(j:m)}}(x) - \alpha^2\frac{(m-1)}{3m^3(m+1)}\sum_{j=1}^{m} E_{X_{(j:m)}}^2[x(1-2F_{X_{(j:m)}}(x))]$ ■

Theorem 2.2. *The pdf for the random variable $X_{[i:m](j)}$ can be written in terms of DF of SRS as*
$f_{X_{[i:m](j)}}(x) = f_{X_{(j:m)}}(x)\Big[1+\alpha C_{m,i}\kappa(i,j,m)\Big([1-2F_X(x)] - A_{j,m}\Big)\Big]$
where the constant

$$
\kappa(i,j,m) = \int f_Y(y)\Big[1-2F_Y(y)\Big]\Big[F_{Y_{[j:m]}}(y)\Big]^{i-1}\Big[1-F_{Y_{[j:m]}}(y)\Big]^{m-i} dy.
$$

3 Estimating Model parameters

In this section, we improve estimating parameters of MTBD distribution that we defined in (1) then, we applied these improved estimators for some specific examples. These examples are: MTBUD and MTBED. As we noted from previous literature, α and β of MTBD were estimated however θ was not. So, in this paper we suggest improving BVRSS to estimate all model parameters.

We start estimating the association parameter α. Similar to Domma and Giardano (2016) estimator in (7), we suggest the following new estimators for α.

Definition 3.1. We develop the following estimators for the association parameter α in (1) by using the first stage BVRSS units $Y_{[i:m]}$ (i.e copies of concomitant of ordered statistics):

1) $\hat{\alpha}_1 = \dfrac{Y_{[i:m]} - \hat{\mu}_Y}{A_{i,m}E_Y[y(1-2F_Y(y,\beta))]}$ at specific i.

2) $\hat{\alpha}_2 = \dfrac{\hat{\mu}_{Y_{[i:m]}} - \hat{\mu}_Y}{A_{i,m}E_Y[y(1-2F_Y(y,\beta))]}$ at specific i where $\hat{\mu}_{Y_{[i:m]}} = \frac{1}{m}\sum_{\omega=1}^{m} Y_{[i:m]\omega}$.

To achieve computing estimators in the above definition, we suggest formulas for $\hat{\mu}_Y$ that introduced in Definition (2.1) or the SRS estimator, $\hat{\mu}_Y = \frac{1}{m}\sum_{i=1}^{m} Y_i$. Hence, in this paper, we consider the SRS estimator.

In the following theorem, we mention properties of estimators in Definition (3.1).

Theorem 3.1. *The estimators in Definition 3.1 are unbiased with variances:*
$$var(\hat{\alpha}_1) = \frac{var_{Y_{[i:m]}}(y) + \frac{1}{m}var_Y(y)}{A^2_{i,m}E^2_Y[y(1-2F_Y(y))]} \text{ and}$$
$$var(\hat{\alpha}_2) = \frac{\frac{1}{m}[var_{Y_{[i:m]}}(y) + var_Y(y)]}{A^2_{i,m}E^2_Y[y(1-2F(y))]}, \text{ respectively.}$$

Proof: We prove this theorem when the SRS average; $\hat{\mu}_Y = \frac{1}{m}\sum_{i=1}^{m} Y_i$; is assumed to define $\hat{\alpha}_1$ and $\hat{\alpha}_2$. We prove the first part of this theorem while the second part can be proved in a similar way.

Proof unbiasedness property of the first estimator, $\hat{\alpha}_1$, can be achieved by noting that the parameter α can be rewritten as $\alpha = \frac{\mu_{Y_{[j:m]}} - \mu_Y}{A_{j,m}E_Y[y(1-2F_Y(y,\beta))]}$, from part 1 of Lemma (2.1). This simply leads to write $E(\hat{\alpha}_1) = \alpha$.

Likewise, proof variance property of $\hat{\alpha}_1$ is straightforward.

Similar proofs can be done when other estimators of μ_Y in Definition (2.1) are used to define $\hat{\alpha}_1$ and $\hat{\alpha}_2$. Considerably, when assuming $\hat{\mu}_Y = \frac{1}{m}\sum_{j=1}^{m} Y_{[j:m]}$, the proof can be achieved by using Lemma (2.1) part 2) while when assuming $\hat{\mu}_Y = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} Y_{(i:m)[j]}$, the proof can be achieved using Theorem (2.1) part 2) ■

Definition 3.2. Using the second stage BVRSS units, we define the following estimators for the association parameter of MTBD:

1) $\hat{\alpha}_3 = \frac{X_{[m-s+1:m](j)} - X_{[s:m](j)}}{2A_{m-s+1,m}E_{X_{(j)}}[x(1-2F_{X_{(j)}}(x))]}$ at specific j and s where s was defined under (7).

2) $\hat{\alpha}_4 = \frac{\hat{\mu}_{X_{[i:m](j)}} - \hat{\mu}_{X_{(j:m)}}}{A_{m-i+1,m}E_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))]}$ at specific i and j where $\hat{\mu}_{X_{(j:m)}} = \frac{1}{m}\sum_{\omega=1}^{m} X_{(j:m)\omega}$ and $\hat{\mu}_{X_{[i:m](j)}} = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} X_{[i:m](j)}$.

Theorem 3.2.

1) The estimator in Definition (3.2), *part 1), is unbiased with variance:*
$$var(\hat{\alpha}_3) = \frac{var_{X_{(j:m)}}(x) - \alpha^2 A^2_{s,m}E^2_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))]}{2A^2_{s,m}E^2_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))]}.$$

2) The estimator in Definition (3.2), *part 2), is unbiased with variance:*
$$var(\hat{\alpha}_4) = \frac{\frac{1}{m^2}[var_{X_{[i:m](j)}}(x) - var_{X_{(j:m)}}(x)]}{A^2_{i,m}E^2_{X_{(j:m)}}[x(1-2F_{X_{(j:m)}}(x))]}.$$

Proof of this theorem is straightforward ■

In the following section, we continue estimating other model parameters that are θ and β. We propose two distribution examples to achieve estimation which are MTBUD and MTBED. These two distributions were defined in (2) and (3) respectively.

3.1 Estimating parameters for specific examples

Scaria and Nair (1999) estimated α and β for MTBD and proved their properties. For MTBUD, they proposed the following unbiased estimators for β:

$$\hat{\beta} = Y_{[m-s+1:m]} + Y_{[s:m]} \tag{8}$$

at specific s with variance $var(\hat{\beta}_U) = \frac{\beta^2}{6}\left(1 - \frac{1}{3}\alpha^2 A_{s,m}^2\right)$. They also proposed a "quick estimator" for α which was originally defined by David and Nagaraja (2003).

For MTBED, Scaria and Nair (1999) proposed the following unbiased estimator for β:

$$\hat{\beta} = \frac{Y_{[m-s+1:m]} + Y_{[s:m]}}{2} \tag{9}$$

at specific s with variance $var(\hat{\beta}_U) = \frac{\beta^2}{2}\left(1 - \frac{1}{4}\alpha^2 A_{s,m}^2\right)$ and they also gave a quick estimator for α. Note that both estimators in (8) and (9) depend on concomitant of ordered statistics.

Next, our purpose is to estimate θ and β for MTBUD and MTBED densities as well as summarize their properties. We improve estimation by using first and second stage samples of BVRSS. It is worth mentioning that our developed estimators of α above are still available for these specific distributions. For the MTBUD specific density, we suggest the following estimators for β:

For MTBUD,

$$\hat{\beta}_U^{(1)} = \frac{2}{m}\sum_{j=1}^{m} Y_{[j:m]} \tag{10}$$

and

$$\hat{\beta}_U^{(2)} = \frac{2}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{m} Y_{(i:m)[j]}. \tag{11}$$

Lemma 3.1. *The estimators in* (10) *and* (11) *are unbiased with variances:*
$var(\hat{\beta}_U^{(1)}) = \frac{\beta^2}{3m}\left[1 - \frac{(m-1)}{9(m+1)}\alpha^2\right]$ *and*
$var(\hat{\beta}_U^{(2)}) = 4\left[\frac{\beta^2}{12m^2} - \frac{1}{m^4}\sum_{j=1}^{m}\sum_{i=1}^{m}(\mu_{Y_{[i:m](j)}} - \frac{\beta}{2})^2\right]$ *respectively.*

Proof unbiasedness property of these estimators is straightforward while proof variances can be concluded from Lemma (2.1) and part 2) of Result (2.2), respectively ■

Parallel, we consider second stage BVRSS samples to estimate θ for MTBUD as follows:

$$\hat{\theta}_U^{(1)} = \frac{1}{m}\sum_{j=1}^{m}\left[X_{[s:m](j)} + X_{[m-s+1:m](j)}\right] \tag{12}$$

at any suitable s that similarly defined as in (7).

$$\hat{\theta}_U^{(2)} = \frac{2}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{m} X_{[i:m](j)}. \tag{13}$$

Lemma 3.2. *The estimators in* (12) *and* (13) *are unbiased with variances:*
$var(\hat{\theta}_U^{(1)}) = \frac{\theta^2}{6m}\left[\frac{2}{(m+1)} - \alpha^2 A_{s,m}^2 \frac{12}{m}\sum_{j=1}^{m}\left[\frac{j}{(m+1)} - 2C_{m,j}\sum_{k=j}^{m}\binom{m}{k}\frac{1}{C_{2m,k+j}}\frac{(k+j)}{(2m+1)}\right]^2\right]$
and
$var(\hat{\theta}_U^{(2)}) = \frac{\theta^2}{3m^2(m+1)}\left[1 - \alpha^2 \frac{2(m-1)}{3m}\sum_{j=1}^{m}\left[\frac{j}{(m+1)} - 2C_{m,j}\sum_{k=j}^{m}\binom{m}{k}\frac{1}{C_{2m,k+j}}\frac{(k+j)}{(2m+1)}\right]^2\right]$,
respectively.

Proof unbiasedness property of these estimators is straightforward while proof variances can be achieved depending on Theorem (2.1) and using the following remark:
$E_{X_{(i)}}(x) = \mu_{X_{(i)}}(x) = \frac{j\theta}{(m+1)}$ and $var_{X_{(i)}} = \frac{j(m-j+1)\theta^2}{(m+1)^2(m+2)}$ ■

Analogously, we can develop estimating θ and β for MTBED. The following estimators are unbiased for β:

$$\hat{\beta}_{\exp}^{(1)} = \frac{1}{m}\sum_{j=1}^{m} Y_{[j:m]} \tag{14}$$

and

$$\hat{\beta}_{\exp}^{(2)} = \frac{1}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{m} Y_{(i:m)[j]} \tag{15}$$

where their variances are
$var(\hat{\beta}_{\exp}^{(1)}) = \dfrac{\beta^2}{m}\left[1 - \dfrac{(m-1)}{12(m+1)}\alpha^2\right]$ and
$var(\hat{\beta}_{\exp}^{(2)}) = \dfrac{\beta^2}{m^2} - \dfrac{1}{m^4}\displaystyle\sum_{j=1}^{m}\sum_{i=1}^{m}(\mu_{Y_{(i:m)[j]}} - \beta)^2.$

The following estimators are unbiased for θ

$$\hat{\theta}_{\exp}^{(1)} = \frac{\sum_{j=1}^{m}\left[X_{[s:m](j)} + X_{[m-s+1:m](j)}\right]}{2m} \tag{16}$$

$$\hat{\theta}_{\exp}^{(2)} = \frac{1}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{m} X_{[i:m](j)} \tag{17}$$

where their variances are

$$var(\hat{\theta}_{\exp}^{(1)}) = \frac{\theta^2}{2m^2}\Bigg(\sum_{j=1}^{m}\sum_{k=1}^{j}\frac{1}{(m-k+1)^2} - \alpha^2 A_{s,m}^2\sum_{j=1}^{m}\left[\sum_{k=1}^{j}\frac{1}{(m-k+1)} - 2C_{m,j}\sum_{k=j}^{m}\binom{m}{k}\frac{1}{C_{2m,k+j}}\sum_{i=1}^{k+j}\frac{1}{(2m-i+1)}\right]^2\Bigg)$$

and

$$var(\hat{\theta}^{(2)}_{\exp}) = \frac{\theta^2}{m^3}\left(\frac{1}{m}\sum_{j=1}^{m}\sum_{k=1}^{j}\frac{1}{(m-k+1)^2}\right.$$
$$\left. -\alpha^2\frac{(m-1)}{3m(m+1)}\sum_{j=1}^{m}\left[\sum_{k=1}^{j}\frac{1}{(m-k+1)} - 2C_{m,j}\sum_{k=j}^{m}\begin{pmatrix} m \\ k \end{pmatrix}\frac{1}{C_{2m,k+j}}\sum_{i=1}^{k+j}\frac{1}{(2m-i+1)}\right]^2\right).$$

The two variances can be proved using the properties: $E_{X_{(i)}}(x) = \theta\sum_{k=1}^{j}\frac{1}{(m-k+1)}$ and $var_{X_{(i)}}(x) = \theta^2\sum_{k=1}^{j}\frac{1}{(m-k+1)}$.

4 Comparing estimators via artificial data

In this section, we illustrated simulation studies to show performance of our developed estimators. We generated data from MTBUD and MTBED, particularly. A general algorithm to yield artificial samples from these densities can be found in (Balakrishnan and Lai (2009)), page 50.

In this paper, we compared our estimators with a few estimators from previous research. Main concepts we used to achieve comparisons are: Average Relative Estimate "ARE" and Relative Efficiency "RE". ARE shows how an estimator is close to the parameter's true value. We define ARE for a sequence of iterated estimators $\hat{\delta}_{1,m}$, $\hat{\delta}_{2,m}$, ..., $\hat{\delta}_{r,m}$ at a true value δ_0 as $ARE(\delta_0) = \frac{1}{r}\sum_{i=1}^{r}\frac{\hat{\delta}_{i,m}}{\delta_0}$ where m is the sample size.

The second concept to achieve comparison is RE which compares quality of two estimators. We define RE to compare the two unbiased estimators, $\hat{\delta}_{1,m}$ and $\hat{\delta}_{2,m}$ say, at specific sample size m by $RE(\hat{\delta}_{2,m}, \hat{\delta}_{1,m}) = \frac{var(\hat{\delta}_{1,m})}{var(\hat{\delta}_{2,m})}$. In some cases, exact variance is intractable so, estimated variance can be used to calculate RE.

In all of our simulation studies, we performed Monte Carlo runs with 10000 iterations to gain estimation. We choose $|\alpha| = 0.25, 0.50, 0.75$ and 0.90 while we choose the sample size as $m = 3, 5, 7$ and 9. Parameters we concern to estimate are α, β and θ for MTBD that defined in model (1). Also, population means μ_X and μ_Y are under investigation.

We considered the estimators $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ and $\hat{\alpha}_4$ that were defined in Section(3) to estimate α. Then we compared quality of these estimators with the estimator defined in (7) using RE principle. Similarly, we used $\hat{\beta}^{(1)}$ and $\hat{\beta}^{(2)}$ to estimate β then efficiency of these estimators compared with the estimator in (8) for MTBUD and compared with the estimator in (9) for MTBED. To estimate θ, we used $\hat{\theta}^{(1)}$ and $\hat{\theta}^{(2)}$ however, since there is no estimators in previous research, we presented variance for estimators.

Remarkably, through simulation studies we run to estimate α where one or more specific ranks s, i or j need to be selected, we found that assuming another values was not change the accuracy of the produced estimators. Therefore, one example was reported for each estimator. Simultaneously, it's recommended to select s appropriately. For example, when assuming $m = 3$, selecting $s = 2$ is prohibited.

We demonstrated simulation runs under two specific examples that are MTBUD and MTBED. The following two sub-sections consider these two distributions respectively.

4.1 Estimating parameters for MTBUD

The proposed distribution to perform simulation in this particular part of study is MTBUD. After generating samples, quality of our developed estimators was investigated. Table (1) shows accuracy of the estimators by computing ARE principle. Even though estimators are either over or under estimate, they still close to their true values. Furthermore, this table shows the accuracy is not affected by varying value of m.

Table 1: Average Relative Estimate for estimated parameters compared with exact value when the density is MTBUD at $s = 1, i = 1$ and $j = m$.

m	$\lvert\alpha\rvert$	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\beta}_{\mathrm{U}}^{(1)}$	$\widehat{\beta}_U^{(2)}$	$\widehat{\theta}_U^{(1)}$	$\widehat{\theta}_U^{(2)}$
3	0.25	1.2331	1.0675	1.1318	1.2365	0.9957	0.9951	1.0944	0.9910
5	0.50	1.0007	0.9798	0.9784	1.0881	0.9975	0.9993	1.0616	0.9931
7	0.75	0.9718	1.0080	0.9760	1.0790	1.0007	0.9980	0.9994	1.0288
9	0.90	0.9928	0.9999	0.9899	1.0815	0.9992	0.9978	0.9991	0.9800

Our goal now is to calculate RE for α estimators compared to the estimator defined in (7) when the underlying distribution is MTBUD. So, variance of each estimator is required. Using Theorem (3.1) , we can find that:
$var(\hat{\alpha}_1) = \dfrac{3 - \alpha^2 A_{i,m}^2 + \frac{3}{m}}{A_{i,m}^2}$ and $var(\hat{\alpha}_2) = \dfrac{\frac{1}{m}(3 - \alpha^2 A_{i,m}^2) + \frac{3}{m}}{A_{i,m}^2}$. Since the difficulty in computing exact $var(\hat{\alpha}_3)$ and $var(\hat{\alpha}_4)$, we used their estimated values. These variances were compared with $var(\hat{\alpha}) = \dfrac{\frac{1}{12} - \alpha^2 A_{s,m}^2 (\frac{1}{6})^2}{2A_{s,m}^2 (\frac{1}{6})^2}$.

Table (2) summarizes RE of the above estimators with respect to $\hat{\alpha}$. It shows some exact and simulated RE. It can be noted that all estimators are efficient but with one less efficient estimator that is $\hat{\alpha}_1$. However, we can increase efficiency by increasing m at fixed α.

Table (3) presents RE for $\hat{\beta}_U^{(1)}$ and $\hat{\beta}_U^{(2)}$ with respect to $\hat{\beta}_U$ and variances for $\hat{\theta}_U^{(1)}$ and $\hat{\theta}_U^{(2)}$. As noted in this table, estimators we improved in this paper are more efficient. However, we can increase efficiency by increasing m at fixed α.

Efficiency of estimators was not affected by changing values of parameters θ and β. This was seen in exact efficiency formulas which are free of these parameters as well as were clearly seen in the simulated outputs. Nevertheless, efficiency was significantly affected by changing values of the rank s when using $\widehat{\theta}_U^{(1)}$.

To see how exact variance of $\widehat{\theta}_U^{(1)}$ vary at different values of s, with fixed m, we give general variance formula at $m = 2$:
$var(\widehat{\theta}_U^{(1)}) = \frac{1}{4}(\frac{2\theta^2}{9} - \frac{16(3-2s)^2\alpha^2\theta^2}{2025})$. It can be noted that the variance increases as s increases.

Table 2: Relative efficiency for $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ and $\hat{\alpha}_4$ with respect to $\hat{\alpha}$ for MTBUD with $s = 1$ and $i = 1$.

m	$\|\alpha\|$	RE($\hat{\alpha}_1, \hat{\alpha}$) Exact	Estimated	RE($\hat{\alpha}_2, \hat{\alpha}$) Exact	Estimated	RE($\hat{\alpha}_3, \hat{\alpha}$) Estimated	RE($\hat{\alpha}_4, \hat{\alpha}$) Estimated
5	0.25	0.4160	0.4159	1.2442	1.2510	1.7438	2.6952
	0.75	0.4104	0.4098	1.1957	1.1999	1.6951	2.3954
	0.95	0.4062	0.4069	1.1605	1.5897	1.6789	2.1019
7	0.25	0.4369	0.4355	1.7397	1.7641	2.2178	3.9152
	0.75	0.4311	0.4297	1.6526	1.6492	2.1081	3.4406
	0.95	0.4266	0.4109	1.5883	1.5783	2.0426	3.1337
9	0.25	0.4494	0.4519	2.2349	2.2400	2.8125	4.5262
	0.75	0.4439	0.4412	2.1064	2.1108	2.6791	4.3189
	0.95	0.4395	0.4352	2.0103	2.0094	2.4803	4.1623

Our last goal in this section is to investigate efficiency of the estimated population means $\mu_X^{(1)}, \mu_X^{(2)}, \mu_Y^{(1)}$ and $\mu_Y^{(2)}$ with respect to SRS estimated population means .

At proposing MTBUD distribution, it is straightforward to prove that exact efficiencies for $\widehat{\mu}_X^{(1)}$ (when using ordered statistics sampling units) and $\widehat{\mu}_Y^{(1)}$ (when using concomitant sampling units)are:

$\mathrm{RE}(\widehat{\mu}_{\mathrm{X}}^{(1)}, \widehat{\mu}_{\mathrm{X}}) = \frac{1}{1-\frac{\mathrm{m}-1}{\mathrm{m}+1}}$ where the produced efficiency depends on the sample size m. Particularly, for $m = 3$ we have RE = 2 while for $m = 5$ gives RE = 3 and for $m = 7$ gives RE = 4.

Although $\mathrm{RE}(\widehat{\mu}_Y^{(1)}, \widehat{\mu}_Y) = \frac{1}{1-\alpha^2 \frac{\mathrm{m}-1}{9(\mathrm{m}+1)}}$. It can be noted that the minimum value of RE is always 1 by assuming α= 0 while the maximum value depends on value of m and taking $|\alpha| = 1$. For m= 3 we have $RE(\mu_Y^{(1)}, \mu_Y) \leq 18/17 = 1.0588$, for m =5 we get $RE(\mu_Y^{(1)}, \mu_Y) \leq 27/25 = 1.08$ and for m =7 we have $RE(\mu_Y^{(1)}, \mu_Y) \leq 12/11 = 1.0909$.

Table (4) reports values of $\mathrm{RE}(\widehat{\mu}_X^{(2)}, \widehat{\mu}_X)$ with respect to SRS population mean $\widehat{\mu}_X$ for some particular values of m and α. It can be realised that the relative efficiency increases by increasing one or both of m and α.

4.2 Estimating parameters for MTBED

The proposed distribution to perform simulation in this particular part of study is MTBED. After generating samples, quality of our developed estimators was investigated. In Table (5), we compared estimators with arbitrarily selected true values to compute ARE. This table can show how our estimators are close to their correspondence true values.

Table (5) reports ARE for estimated parameters compared with exact value when the density is MTBED at s =1, i = 1 and j= m.

Table 3: Relative efficiency for $\widehat{\beta}_U^{(1)}$ and $\widehat{\beta}_U^{(1)}$ with respect to $\widehat{\beta}_U$ and variances of $\widehat{\theta}_U^{(1)}$ and $\widehat{\theta}_U^{(2)}$. The proposed parameters $\theta = 1$, $\beta = 1$ and $s = 1$ to compute $\widehat{\theta}_U^{(1)}$.

m	$\lvert\alpha\rvert$	RE($\widehat{\beta}_U^{(1)}$, $\widehat{\beta}_U$)		RE($\widehat{\beta}_U^{(2)}$, $\widehat{\beta}_U$)	$Var(\widehat{\theta}_U^{(1)})$	$Var(\widehat{\theta}_U^{(2)})$
		Estimated	Exact	Estimated	Exact	Exact
3	0.25	1.4993	1.4974	1.9701	0.02764	0.01845
	0.75	1.4709	1.4758	1.9488	0.02654	0.01797
	0.90	1.4840	1.4647	1.9401	0.02580	0.01764
5	0.25	2.4468	2.4884	2.9789	0.01101	0.00443
	0.75	2.4136	2.3913	2.8955	0.01024	0.00427
	0.90	2.3705	2.3404	2.5607	0.00971	0.00417
7	0.25	3.4344	3.4771	3.9991	0.00589	0.00169
	0.75	3.2887	3.2887	3.8021	0.00536	0.00162
	0.90	3.1185	3.1833	3.6972	0.00500	000158
9	0.25	4.5045	4.4648	4.9861	0.00366	0.00082
	0.75	4.3052	4.1684	4.7508	0.00328	0.00078
	0.90	3.9811	4.0112	4.5973	0.00303	0.00076

Our goal now is to calculate RE for α estimators compared to the estimator defined in (7) when the underlying distribution is MTBED. So, variance of each estimator is required. Using Theorem (3.1) , we can find that:
$var(\hat{\alpha}_1) = \dfrac{4 - 2\alpha A_{i,m} - \alpha^2 A_{i,m}^2 + \frac{4}{m}}{A_{i,m}^2}$ and $var(\hat{\alpha}_2) = \dfrac{\frac{1}{m}(4 - 2\alpha A_{i,m} - \alpha^2 A_{i,m}^2 + \frac{4}{m})}{A_{i,m}^2}$. Since the difficulty in computing exact $var(\hat{\alpha}_3)$ and $var(\hat{\alpha}_4)$, we used their estimated values to calculate RE. These variances were compared to $var(\hat{\alpha}) = \dfrac{1 - \alpha^2 A_{s,m}^2(\frac{1}{2})^2}{2A_{s,m}^2(\frac{1}{2})^2}$.

Table (6) summarizes RE of these estimators with respect to $\hat{\alpha}$. It shows some exact and simulated RE. It can be noted that our estimators are efficient with one less efficient estimator that is $\hat{\alpha}_1$. However, we can increase efficiency by increasing the sample size m.

Table (7) presents RE of $\hat{\beta_{exp}^{(1)}}$ and $\hat{\beta_{exp}^{(2)}}$ with respect to $\hat{\beta}_{exp}$. Importantly, there is no previous estimators for θ to compute REs so, we show variances of $\hat{\theta^{(1)}}_{exp}$ and $\hat{\theta_{exp}^{(2)}}$.

To see how exact variance of $\widehat{\theta}_{exp}^{(1)}$ changing at different values of s, with fixed m, we give general variance formula at $m = 2$:
$var(\widehat{\theta}_{exp}^{(1)}) = \frac{3\theta^2}{16} - \frac{29(2-2s)^2\alpha^2\theta^2}{5184}$. It can be noted that the variance increases as s increases.

Our last goal from this section is to investigate efficiency of the estimated population

Table 4: Relative efficiency for $\widehat{\mu}_X^{(2)}$ with respect to SRS population mean $\widehat{\mu}_X$ when MTBUD.

m	$\lvert\alpha\rvert$	RE$(\widehat{\mu}_X^{(2)}, \widehat{\mu}_X)$ Exact
3	0.25	2.0066
	0.75	2.0610
	0.95	2.0998
5	0.25	3.0131
	0.75	3.1225
	0.95	3.2016
7	0.25	4.0197
	0.75	4.1847
	0.95	4.3048

Table 5: Average Relative Estimate for estimated parameters compared with exact value when the density is MTBED at $s = 1, i = 1$ and $j = m$

$\boldsymbol{m}$	$\lvert\alpha\rvert$	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\beta}^{(1)}_{exp}$	$\widehat{\beta}^{(2)}_{exp}$	$\widehat{\theta}^{(1)}_{exp}$	$\widehat{\theta}^{(2)}_{exp}$
3	0.25	1.0319	0.9092	0.9615	0.9906	1.0435	1.0062	1.0754	1.0294
5	0.50	1.0219	1.0075	0.9947	0.9159	1.0182	1.0501	1.0595	1.0120
7	0.75	0.9808	1.0055	1.0003	0.9079	1.0272	1.0174	1.0397	1.0663
9	0.90	0.9931	0.9998	0.9991	0.9129	1.0457	0.9913	0.9976	1.0126

means $\mu_X^{(1)}, \mu_X^{(2)}, \mu_Y^{(1)}$ and $\mu_Y^{(2)}$ with respect to SRS estimated population means .

At proposing MTBED distribution, the efficiency for $\widehat{\mu}_X^{(1)}$ (when using ordered statistics sampling units) and $\widehat{\mu}_Y^{(1)}$ (when using concomitant sampling units) can be written as:

RE$(\widehat{\mu}_X^{(1)}, \widehat{\mu}_X) = \frac{1}{1-\frac{1}{m}\sum_{i=1}^{m}(\sum_{k=1}^{i}\frac{1}{m-k+1}-1)^2}$ where it depends only on the sample size m. Specific examples to compute the efficiency at selected m is given next. For $m = 3$ we have RE $= \frac{18}{11} = 1.6363$ while for $m = 5$ gives RE $= \frac{300}{137}$ and for $m = 7$ gives RE $= \frac{980}{363} = 2.6997$.

Similarly, we can investigate efficiency of $\widehat{\mu}_Y^{(1)}$ which is simply can be written as RE$(\widehat{\mu}_Y^{(1)}, \widehat{\mu}_Y) = \frac{1}{1-\alpha^2\frac{\mathrm{m}-1}{9(\mathrm{m}+1)}}$. It can be noted that the minimum value of RE is always 1 by assuming α= 0 while the maximum value depends on value of m and taking

Table 6: Relative efficiency for $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ and $\hat{\alpha}_4$ with respect to $\hat{\alpha}$ for MTBED with $s = 1$ and $i = 1$.

m	$\|\alpha\|$	RE$(\hat{\alpha}_1, \hat{\alpha})$ Exact	Estimated	RE$(\hat{\alpha}_2, \hat{\alpha})$ Exact	Estimated	RE$(\hat{\alpha}_3, \hat{\alpha})$ Estimated	RE$(\hat{\alpha}_4, \hat{\alpha})$ Estimated
5	0.25	0.4474	0.4401	1.3000	1.3109	1.7609	2.3109
	0.75	0.5282	0.5349	1.3889	1.3774	1.7856	2.4598
	0.95	0.5745	0.5589	1.4207	1.4124	1.8027	2.7525
7	0.25	0.4764	0.4631	1.8284	1.8115	2.1015	3.1537
	0.75	0.5884	0.5860	1.9658	1.9566	2.1687	3.4628
	0.95	0.6617	0.6704	2.0146	2.1735	2.2904	3.9485
9	0.25	0.4945	0.4886	2.3571	2.3382	2.6803	4.1901
	0.75	0.6310	0.6324	2.5425	2.5841	2.9179	4.3798
	0.95	0.7291	0.7361	2.6092	2.5908	3.0081	4.5571

$|\alpha| = 1$. For example, if m= 3 we get $RE(\mu_Y^{(1)}, \mu_Y) \leq \frac{24}{23} = 1.0435$, for m =5 we get $RE(\mu_Y^{(1)}, \mu_Y) \leq \frac{18}{17} = 1.0588$ and for m =7 we have $RE(\mu_Y^{(1)}, \mu_Y) \leq \frac{16}{15} = 1.0667$.

Table (8) reports values of RE$(\widehat{\mu}_X^{(2)}, \widehat{\mu}_X)$ for some particular values of m and α. It can be realised that the relative efficiency increases by increasing one of m and α or both.

5 Conclusions and Discussion

A main benefit of RSS, as well as BVRSS, is originally to gain more information from ordered observations after quantification rather than SRS. This research gained an extra advantage from a modification on stages of BVRSS where the first stage units (which include copies of concomitant of ordered statistics) were used to achieve estimation similarly as the second stage units. An example can be stated here is when estimating α by using $\hat{\alpha}_4$ where copies of concomitant of ordered statistics were used to compute $\hat{\mu}_{X_{(j:m)}}$.

Efficiency of our estimators was not affected by changing values of distribution parameters; θ and β. We were seen this in exact efficiency formulas which are free of these parameters as well as were clearly seen in the simulated outputs. Additionally, efficiency of the estimators $\hat{\alpha}_3$ and $\hat{\theta}^{(1)}$ attains its maxima when $s = 1$ since this rank gives the minimum variance.

In both distribution examples that considered in this paper, we found that $\hat{\alpha}_1$ is less efficient than SRS estimator however, it can be noted that this estimator is doing better for MTBED than MTBUD. Also, the efficiency can be improved by increasing m.

It is well known in the literature that maximum RE$(\widehat{\mu}_X^{(1)}, \widehat{\mu}_X)$ (i.e using RSS units vs SRS units) can be achieved if the distribution is uniform and its value is $\frac{1}{1-\frac{m-1}{m+1}}$. In this paper, we proved that maximum RE$(\widehat{\mu}_X^{(1)}, \widehat{\mu}_X)$, if the distribution is exponential, is

Table 7: Relative efficiency for $\hat{\beta}^{(1)}_{exp}$ and $\hat{\beta}^{(2)}_{exp}$ with respect to $\hat{\beta}_{exp}$. The proposed value of s to compute $\hat{\theta^{(1)}}_{exp}$ is 1.

m	$\lvert\alpha\rvert$	RE($\widehat{\beta}^{(1)}_{exp}$, $\widehat{\beta}_{exp}$) Estimated	exact	RE($\widehat{\beta}^{(2)}_{exp}$, $\widehat{\beta}_{exp}$) Estimated	var($\hat{\theta^{(1)}}_{exp}$) Exact	Var($\hat{\theta^{(2)}}_{exp}$) Exact
3	0.25	1.3435	1.4980	1.9663	0.10141	0.06771
	0.75	1.5306	1.4820	1.9357	0.09788	0.06614
	0.90	1.4928	1.4738	1.9291	0.09613	0.06536
5	0.25	2.2378	2.4913	2.9686	0.04530	0.01809
	0.75	2.5533	2.4194	2.8832	0.04240	0.01761
	0.90	2.6498	2.3822	2.5557	0.04097	0.01733
7	0.25	3.1871	3.4828	3.8997	0.02619	0.00753
	0.75	3.4783	3.3406	3.7792	0.02403	0.00725
	0.90	3.7106	3.2667	3.6681	0.02296	0.00712
9	0.25	4.3203	4.4736	4.9683	0.01726	0.00386
	0.75	4.5319	4.2546	4.7013	0.01562	0.00371
	0.90	4.6730	4.1404	4.5558	0.01481	0.00364

equal to $\dfrac{1}{1-\frac{1}{m}\sum_{i=1}^{m}\left(\sum_{k=1}^{i}\frac{1}{m-k+1}-1\right)^2}$.

Moreover, for the concomitant of ordered statistics for MTBD, we showed that maximum RE$(\widehat{\mu}^{(1)}_Y, \widehat{\mu}_Y)$ is equal to $\frac{1}{1-\alpha^2\frac{m-1}{9(m+1)}}$ if the marginal distribution is uniform and is equal to $\frac{1}{1-\alpha^2\frac{m-1}{12(m+1)}}$ if the marginal distribution is exponential.

Some researchers pay attention on proving the relationship between the association parameter in MTBD and the population correlation coefficient $\boldsymbol{\rho}$. See for example, Schucany et al. (1978) and Scaria and Nair (1999). Specifically, for MTBUD the relation is $\rho = \alpha/3$ and for MTBED the relation is $\rho = \alpha/4$. Therefore, we can produce many estimators for ρ by substituting estimators of α that we derived in Definition (3.1) or Definition (3.2) on these relations. Thus, a proposed estimator for MTBUD is $\widehat{\rho} = \widehat{\alpha}/3$ and for MTBED is $\widehat{\rho} = \widehat{\alpha}/4$.

Acknowledgement

The authors are very thankful to the referees who gave crucial notes and suggestions which improved the earlier drafts of this paper.

Table 8: Relative efficiency for $\widehat{\mu}_X^{(2)}$ with respect to SRS population mean $\widehat{\mu}_X$ when MTBED.

$\boldsymbol{m}$	$\lvert\alpha\rvert$	RE($\widehat{\mu}_X^{(2)}, \widehat{\mu}_X$) Exact
3	0.25	1.6411
	0.75	1.6800
	0.95	1.7076
5	0.25	2.1985
	0.75	2.2709
	0.95	2.3230
7	0.25	2.7120
	0.75	2.8146
	0.95	2.8890

References

Abo-Eleneen, Z. and Nagaraja, H. (2002). Fisher information in an order statistic and its concomitant. *Ann. Inst. Statist. Math.*, 54(3):667–680.

Al Kadiri, M. and Migdadi, M. (2018). Estimating morgenstern type bivariate association parameter using a modified maximum likelihood method. *accepted for publication*, 0.

Al-Saleh, M. and Al Kadiri, M. A. (2000). Double-ranked set sampling. *Statistics and Probability Letters*, 48:205–212.

Al-Saleh, M. and Al-omari, A. (2002). Multistage ranked set sampling. *Journal of Statistical Planning and Inference*, 102(2):273–286.

Al-Saleh, M. and Zheng, G. (2002). Estimation of bivariate characteristics using ranked set sampling. *Australian and New Zealand Journal of Statistic*, 44:221–232.

Balakrishnan, N. and Lai, C. (2009). *Continuous Bivariate Distributions. 2nd ed.* Springer.

Chacko, M. and Thomas, Y. (2008). Estimation of a parameter of morgenstern type bivariate exponential by ranked set sampling. *Annals of the Institute of Statistical Mathematics*, 60:273–300.

David, H. and Nagaraja, H. (2003). *Order statistics.* New York: Wiley.

Domma, F. and Giardano, S. (2016). Concomitants of m-generalized order statistics from generalized farlie–gumbel–morgenstern distribution family. *Journal of Computational and Applied Mathematics*, 294:413–435.

Ehsanullah, M., Nevzorov, V., and Shakil, M. (2013). *An Introduction to Order Statis-*

tics. Atlantis Press.

Genest, C., K, G., and L, R. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82:543–552.

Mclntyre, G. (1952). A method for unbiased selective sampling using ranked sets. *Australian Journal of Agricultural Research*, 3:385–390.

Morgenstern, M. (1956). Simple examples of two-dimensional distributions. *Statistik*, 8:234–235.

Scaria, J. and Nair, N. (1999). On concomitants of order statistics from morgenstern family. *Biometrical Journal*, 41:483–489.

Schucany, W., Parr, W., and Boyer, J. (1978). Correlation structure in farile –gumbel–morgenstern distributions. *Biometrika*, 65:650–653.

Singh, H. and Mehta, V. (2016). Improved estimation of scale parameter of morgenstern type bivariate uniform distribution using ranked set sampling. *Communications in Statistics – Theory and Methods*, 45(5):1466–1476.

Stefanescu, C. and Turnbull, B. (2009). Likelihood inference for exchangeable continuous data with covariates and varying cluster sizes; use of the farlie–gumbel–morgenstern mode. *Statistical Methodology*, 6:503–512.

Tahmasebi, S. and Jafari, A. (2014). Estimators for the parameter mean of morgenstern type bivariate generalized exponential distribution using ranked set sampling. *Statistics and Operations Research Transactions*, 38:1–19.

Takahasi, K. and Wakimoto, K. (1968). On unbiased estimates of the population mean based on the sample stratified by means of ordering. *Ann. Inst. Statist. Math*, 20:421–428.

Wolfe, D. (2012). Ranked set sampling: Its relevance and impact on statistical inference. *ISRN Probability and Statistics*, 2012:32.