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Comparison of regression models under multicollinearity

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Multicollinearity is a major problem in linear regression analysis and several methods exists in the literature to deal with the same. Ridge regression is one of the most popular methods to overcome the problem followed by Generalized Ridge Regression (GRR) and Directed Ridge Regression (DRR). However, there exist many computational issues in using the above methods. Partial Ridge Regression (PRR) method is a computationally viable approach by selectively adjusting the ridge constants using the cutoff criteria. In this paper, the performance of the Partial Ridge Regression approach has been evaluated through a simulation study based on the mean squared error (MSE) criterion. Comparing with other methods of ridge regression, the study indicates that the Partial ridge regression by cutoff criteria performs better than the existing methods.

keywords: Linear regression, Multicollinearity, Ridge Estimator, Generalized Ridge, Directed Ridge, Partial Ridge, Least Square Estimator, Mean Squared Error, Signal to Noise, Proportion of Replication.

1 Introduction

Multiple linear regression model usually assumes independence of explanatory variables. However, practically, strong linear relationships may exist among explanatory variables and thereby assumption of independence gets violated giving rise to the problem of multicollinearity. It is not possible to estimate the distinct effects along with the standard

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error of individual variables in the regression analysis. Thus, inference and prediction based on regression models are affected disproportionately in the presence of collinearity. There are several methods existing in the literature to solve this problem and ridge regression is one of the most popular methods among them. The linear multiple regression model is given by,

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{1}$$

where $\mathbf{Y}$ is an n x 1 vector of observations, $\boldsymbol{\beta}$ is a p x 1 vector of unknown regression coefficients. $\mathbf{X}$ is an n x p known design matrix of rank p, and $\boldsymbol{\varepsilon}$ is an n x 1 vector of random errors, which is distributed as multivariate normal with mean 0 and covariance matrix $\sigma^2\mathbf{I}$ being an Identity matrix of order n. The OLS estimator of $\boldsymbol{\beta}$ is obtained as

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y} \tag{2}$$

and covariance matrix of $\hat{\boldsymbol{\beta}}$ is $Cov\left(\hat{\boldsymbol{\beta}}\right) = \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}$. The regression model assumes that regressors are nearly independent. In many practical situations regressors are nearly dependent. In that case the matrix $(\mathbf{X}'\mathbf{X})$ becomes ill conditioned, i.e. $det\left(\mathbf{X}'\mathbf{X}\right) \approx 0$). If $(\mathbf{X}'\mathbf{X})$ is ill conditioned, then $\left(\hat{\boldsymbol{\beta}}\right)$ is sensitive to a number of errors and therefore meaningful statistical inferences becomes very difficult for practitioner. To overcome this problem, Hoer and Kennard (1970) suggested a small positive number to be added to diagonal element of the matrix $(\mathbf{X}'\mathbf{X})$. Thus resulting estimators are obtained as,

$$\begin{aligned}\hat{\boldsymbol{\beta}}(k) &= \left(\mathbf{X}'\mathbf{X} + k\mathbf{I}_p\right)^{-1}\mathbf{X}'\mathbf{Y}\\ &= \mathbf{W}\hat{\boldsymbol{\beta}}\end{aligned} \tag{3}$$

where $\mathbf{W} = \left(\mathbf{I}_p - k\mathbf{C}^{-1}\right)^{-1}\mathbf{X}'\mathbf{Y}$, $k \geq 0$, $\mathbf{C} = \mathbf{X}'\mathbf{X}$, and $\mathbf{I}_p$ is an identity matrix of order p. This is known as the ridge regression estimator. Since the quantity $(\mathbf{X}'\mathbf{X} + k\mathbf{I}_p)$ in (2) is always invertible, there always exist a unique solution for $\hat{\boldsymbol{\beta}}(k)$ (Draper and Smith, 2014). The ridge regression estimator is a biased estimator and for a positive value of k, these estimators provide a smaller MSE compared to the OLS estimator. From (2) we observe that as $k \to \infty$ and $\hat{\boldsymbol{\beta}}(k) \to 0$. Various methods are suggested by many researchers for choosing ridge parameter in ridge regression (Dorugade, 2014) (Ciavolino and Indiveri, 2013).

The choice of k plays an important role in the study of ridge regression. Further, Hoer and Kennard (1970b) have adapted a suitable modification in determining the values of k and proposed Generalized Ridge Regression (GRR). Guilkey and Murphy (1975) have proposed an alternative approach of Directed Ridge Regression (DRR) to determine the values of k in the analysis of linear regression models. Chandrasekhar et al (2016) developed a cut off criteria in choosing the suitable value of k for the analysis under Partial Ridge Regression (PRR). The performance of this approach has been evaluated through a comparative study. The objective of this paper is to investigate the existing methods that are available and to make a comparison among them based on the mean square error.

The present paper details out the various methods of Ridge Regression viz., GRR, DRR, PRR given in Section 2. Section 3 describes the Monte Carlo Simulation. Simulation results are discussed in Section 4. An example has been considered in Section 5. Finally the conclusion is given in Section 6.

2 Ridge Regression Methods

2.1 Generalized Ridge Regression

Hoer and Kennard (1970) proposed an extension of the ridge regression procedure that allows separate biasing parameters k for each regressor known as generalized ridge regression. The discussion on generalized ridge regression approach of estimation is simplified by a suitable orthogonal transformation of regressors. Suppose there exists an orthogonal matrix $\mathbf{D}$ such that $\mathbf{D'CD} = \mathbf{\Lambda}$. Where $\mathbf{\Lambda} = diag[\lambda_1, \lambda_2, ..., \lambda_p]$ contains the eigen values of matrix $\mathbf{C}$. The Orthogonal version of the model is

$$\mathbf{Y} = \mathbf{X}^* \boldsymbol{\alpha} + e \tag{4}$$

where $\mathbf{X}^* = \mathbf{XD}$ and $\boldsymbol{\alpha} = \mathbf{D}'\boldsymbol{\beta}$. Then the generalized ridge regression estimators is given as follows

$$\hat{\boldsymbol{\alpha}}(k) = \left(\mathbf{X}^{*'}\mathbf{X'} * + \mathbf{K}\right) \mathbf{X}^*\mathbf{Y} \tag{5}$$

Where $\mathbf{K} = diag(k_1, k_2, ..., k_p)$, $k_i > 0$ and $\hat{\boldsymbol{\alpha}} = \mathbf{\Lambda}^{-1}\mathbf{X}^*\mathbf{Y}$ is the ordinary least squares (OLS) estimates of $\boldsymbol{\alpha}$. The vector of original parameter estimates is given by using 4 that is,

$$\hat{\boldsymbol{\beta}} = \mathbf{D}\boldsymbol{\alpha} \tag{6}$$

In terms of the original model, the generalized ridge coefficients are

$$\hat{\boldsymbol{\beta}}(k) = \mathbf{D}\hat{\boldsymbol{\alpha}}(k) \tag{7}$$

The mean square error for generalized ridge regression is

$$\begin{aligned} MSE\left(\hat{\boldsymbol{\beta}}(k)\right) = E\left[\left(\hat{\boldsymbol{\beta}}(k) - \boldsymbol{\beta}\right)' \left(\hat{\boldsymbol{\beta}}(k) - \boldsymbol{\beta}\right)\right] \\ E\left[(\hat{\boldsymbol{\alpha}}(k) - \boldsymbol{\alpha})' (\hat{\boldsymbol{\alpha}}(k) - \boldsymbol{\alpha})\right] \end{aligned} \tag{8}$$

It follows from **?** that the value of k_i which minimizes the $MSE\left(\hat{\boldsymbol{\alpha}}(k)\right)$, where,

$$MSE\left(\hat{\boldsymbol{\alpha}}(k)\right) = \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k_i)^2} + \sum_{i=1}^{p} \frac{k_i \alpha_i^2}{(\lambda_i + k_i)^2} \tag{9}$$

when $k_i = \sigma^2/\alpha_i^2$, σ^2 represents the error variance of model (1), α_i is the ith element of $\boldsymbol{\alpha}$. The optimal value of k_i, fully depends on the unknown σ and α_i, and they must be estimated from the observed data. **?** suggested to replace σ^2 and α_i, by their corresponding unbiased estimators. That is,

$$\hat{k}_i = \hat{\sigma}^2 / \hat{\alpha_i}^2 \tag{10}$$

with $\hat{\sigma}^2 = \sum e_i^2/(n-p)$ is the residual mean square estimate, which is an unbiased estimator of σ^2 and α_i is the ith element of $\hat{\boldsymbol{\alpha}}$, which is an unbiased estimator of $\boldsymbol{\alpha}$. The method of GRR is a computationally intensive approach of dealing with multicollinearity. Guilkey and Murphy (1975) suggested that a selective approach of biasing the parameters may be considered than biasing all the parameters $(k_1, k_2, ..., k_p)$.

2.2 Directed Ridge Regression

In this method, shrinking of elements of the parameter vector is restricted to only those coefficients corresponding to small eigen values and the shrinkage is done using generalized ridge. Guilkey and Murphy (1975) suggested that $\mathbf{X}_i$ be defined as small if $\lambda_i < 10^{-c}\lambda_{max}$, where λ_{max} is the largest eigen value of $\mathbf{X'X}$ and c is some arbitrary constant. The smallest eigen values are first identified, followed by a procedure similar to the ridge estimator method and add only a k_i^* added only to properly selected diagonal elements of $\boldsymbol{\Lambda}$.
Let be $\mathbf{K} = diag\,(k_1, k_2, ..., k_p)$ with $k_i^{*(0)} = \hat{\sigma}/\hat{\alpha}_i^2$, the ridge regression estimator is given as follows,

$$\hat{\boldsymbol{\alpha}}(k) = \left(\mathbf{X}^{*\prime}\mathbf{X} + \mathbf{KI}\right)\mathbf{X}^{*}\mathbf{Y} \tag{11}$$

Let $k_i^{*(0)} = 0$ for i such that $\lambda_i \geq 10^{-c}\lambda_{max}$, find the directed ridge estimator is given by repeat the following two steps until stabilization is achieved, by the mth iteration.

1. $\boldsymbol{\alpha}^{*(0)} = \lambda_{K^*}^{-1}\mathbf{X}^{*\prime}\mathbf{Y}$

2. $k_i^{*(1)} = \hat{\sigma}^2/\alpha_i^{*(1)^2}$

Find $\boldsymbol{\beta}^* = \mathbf{D}\boldsymbol{\alpha}^{*(m)}$, the process is still computationally intensive as in GRR but to lesser extent as still a partial set of vectors $(k_1, k_2, ..., k_p)$ need to be obtained.

2.3 Partial Ridge Regression

Chandrasekhar et al (2016) proposed a PRR approach, which involves selectively adjusting the ridge constants associated with highly collinear variables to control instability in the variances of coefficient estimates (Nduka and Ijomah, 2012). In PRR a cutoff criteria method has been adapted to first identify q $(q < p)$ collinear regressors and biasing constant k is added only to the q regressors. In this method, Singular Value Decomposition (SVD) is used for selection of biasing the collinear variables for acquiring competent estimates of the regression coefficients. SVD is a popular dimension reduction technique and its main features are, (i) ill-conditioning in $\mathbf{X}$ is reflected in the size of singular values and small singular values convey near-linear dependency of the regressors

(ii) singular values are the diagonal elements arranged in the order of their size, so as to make partial biasing easier.

Let $\mathbf{X}$ be $n \times p$ matrix and on applying SVD to $\mathbf{X}'\mathbf{X}$, then

$$\mathbf{X}'\mathbf{X} = \mathbf{U}\boldsymbol{\Delta}\mathbf{U}' \tag{12}$$

where Δ is a $(p \times p)$ diagonal matrix whose n diagonal elements are the Eigenvalues λ_j, $j = 1, 2, ..., p$. Following the classical ridge regression as given in (5),

$$\hat{\boldsymbol{\beta}}_R = \left(\mathbf{U}\boldsymbol{\Delta}\mathbf{U}' + k\mathbf{I}\right)\mathbf{X}'\mathbf{Y} \tag{13}$$

$$var\left(\hat{\boldsymbol{\beta}}_R\right) = \left(\mathbf{U}\boldsymbol{\Delta}\mathbf{U}' + k\mathbf{I}\right)\sigma^2 = \begin{bmatrix}\mathbf{U}\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_p\end{bmatrix}\mathbf{U}'\end{bmatrix} + \begin{bmatrix}k_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & k_p\end{bmatrix}\mathbf{I}\sigma^2 \tag{14}$$

As in the case of ridge regression, $0 \le k_i \le 1$, $(i = 1, ..., p)$ a single value of $k_1, k_2, ..., k_p$ is added only to relevant variables having large eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$ and the other values are kept as zero. The eigenvalues of $\mathbf{U}\boldsymbol{\Delta}\mathbf{U}'$ are used to decide cutoff based criteria and scree plot could be an useful tool in deciding the cut-off index m with $m < p$. The ratio of jth eigenvalue to largest eigenvalue, say $\delta = \lambda_j/\lambda_1$ is a useful indicator of near-linear dependence. The cut off value of the order 0.001 (one-thousandth) is an useful measure in deciding cut-off index m. Bias k is then added to only those diagonal elements having an index greater than m. It may be noted that if cut off is $m = 0$ or $k = 0$ then the method is reduced to OLS and if cutoff equal to p then our method is same as classical ridge regression method.

Chandrasekhar et al (2016) has shown that in PRR bias is added only to m out of p regressors. The various bias values of k ranges from 0.001 to 0.512 (Montgomery et al, 2015). In order to attain stability in MSE, the k values are further considered from 0.55 to 11.55. The study has also shown the regression coefficients obtained stability for a partial set of regressors and attains minimum MSE compared to Ridge regression and OLS. However, the efficiency of PRR approach needs to be evaluated in comparison with RR, GRR and DRR methods discussed in the earlier sections.In the next section, to know which method show better performance, all the ridge regression methods given above are evaluated under simulation study.

3 The Monte Carlo Simulation

The aim of this study is to compare the performance of different ridge regression methods and find good ridge regression method. A simulation study has been conducted using R programming language, version 3.3.2 (R Core Team, 2014). Because degree of collinearity among explanatory variables ($\mathbf{X}$) is of central importance, Kibria (2003) is followed for generating $\mathbf{X}$ variables using the following equation

$$x_{ij} = \left(1 - \gamma^2\right)^{1/2} z_{ij} - \gamma z_{ip} \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., p \tag{15}$$

where z_{ij} are independent standard normal pseudo random numbers and γ represents correlation between any two $\mathbf{X}$ variables. These variables are standardized so that $\mathbf{X}'\mathbf{X}$ and $\mathbf{X}'\mathbf{Y}$ are in correlation forms. The n observations for $\mathbf{Y}$ are determined by the following equation

$$y_i = \beta_0 + \beta_1 x_{i2} + ... + \beta_p x_{ip} + e_i \quad i = 1, 2, ..., n \tag{16}$$

with e_i are independent normal $(0, \sigma^2)$ pseudo- random numbers, and β_0 is equal to zero. The sample size n varied between 15 and 25 and the p explanatory variables between 4 and 10. In this study, five values of σ are investigated which are given by 0.01, 0.1, 0.25, 0.5 and 1. For each set of explanatory variables, one choice for the coefficient vectors is considered. Newhouse and Oman (1971) stated that if the mean squared error is a function of β, σ^2 and k if the explanatory variables are fixed, then the MSE is minimized when β is the normalized eigen vectors corresponding to the largest eigen value of $\mathbf{X}'\mathbf{X}$ subject to the constraint that $\beta'\beta = 1$. Hence the coefficients $\beta_1 + \beta_2 + ... + \beta_p$ as normalized eigen vectors corresponding to the largest eigen values of $\mathbf{X}'\mathbf{X}$ matrix is selected so that $\beta'\beta = 1$. Thus, for n, p, γ, β, ρ^2 and σ, sets of $\mathbf{X}$ variables are generated. Then the experiment was repeated 2000 times by generating new error terms. Different ridge regression methods are applied for the values of k and average mean square error are estimated and presented in the Appendix. A relationship between σ^2 and the signal to noise ratio is considered as

$$\rho^2 = \beta'\beta/\sigma^2 \tag{17}$$

The values of ρ^2 corresponding to σ are 10000, 100, 16, 4 and 1 respectively. For each replicate r $(r = 1, 2, ..., 2000)$ the ridge estimators were calculated using

$$\hat{\boldsymbol{\alpha}}(k) = \left(\boldsymbol{\Lambda} + \hat{k} \times \mathbf{I}_p\right) \mathbf{X}^*\mathbf{Y} \tag{18}$$

Then the MSE for the estimators are calculated as

$$MSE\left(\hat{\boldsymbol{\alpha}}(k)\right) = \frac{1}{2000} \sum_{r=1}^{2000} \left(\hat{\boldsymbol{\alpha}}_r - \boldsymbol{\alpha}\right)' \left(\hat{\boldsymbol{\alpha}}_r - \boldsymbol{\alpha}\right) \tag{19}$$

The simulated mean squared errors are summarized in the Appendix A. The proportion of replication for which the least squares estimators produced a smaller MSE than the other ridge regression methods were calculated. In the next section the results of the different methods are discussed in tables and discussed about the results.

4 Analysis of the Mean square error for different methods

For the given values of ρ^2 $(< 10,000)$, all the ridge regression methods have a smaller mean square error than the ordinary least squares (see Appendix A). The degree of correlation, the number of observations and the number of explanatory variables are the factors that influence the MSE. It is observed that, when the value of ρ^2 increases, mean square error decreases. For the lower k values (0.001 to 0.512), the MSE values obtained under PRR and RR methods perform equally well. However, a moderate increase is observed in the MSE values of PRR compared to RR for higher values of k (0.55 to 1.05). Considering all the methods, PRR and RR performs equally well for all values of ρ^2 when $k = 0.001$ to 0.256. When $k = 0.055$ to 1.05 the PRR MSE value is having a moderate increased value with RR for the values of $n = 15$, $p = 4$, $\gamma = 0.7, 0.8, 0.9, 0.99$ and $k = 0.001$ to 0.512 and increases gradually for $k = 0.55$ to 1.05. But for $n = 25$, $p = 10$ and $\gamma = 0.7, 0.8, 0.9, 0.99$ the MSE values are equal till $k = 0.001$ to 0.512 and moderate increase when $k = 0.55$ to 1.05. By fixing γ and p and increasing the number of observations has a better impact on all the methods, especially in PRR compared to GRR ,DRR for $\rho^2 = 10000, 100, 16, 4$. The mean square error shows a good sign of lower value in the case of $n = 15, 25$, $p = 4, 10$ and $\gamma = 0.99$. Holding γ fixed and increasing the values of p and n, the mean square error is lower for PRR at $k = 0.55$ to 1.05 while $\rho^2 = 10000, 100, 16$ at $n = 25$, $p = 10$ and $\gamma = 0.99$. However MSE values are higher when $\rho^2 = 4, 1$. But in the methods GRR and DRR, the mean square values are higher when n and p increases. The stability of the mean square error is obtained for different k values at $\rho^2 = 100, 16, 4, 1$. The results are presented in Table 5 to16 for $n = 15$, $p = 4$ and $\gamma = 0.7, 0.8, 0.9, 0.99$ respectively and results are presented for and stability of the mean square error for $n = 25$, $p = 10$ and $\gamma = 0.7, 0.8, 0.9, 0.99$ respectively in the Tables 17 to 22. It has been observed from the above tables when γ increases mean square values are stabilized. The k values range between 0.064 to 1.55 for lower values of n and p, 0.55 to 10.55 for higher values of n and p.

In Table 5-28 the simulated MSE are presented for different sample size n, p and γ. The results from these tables are summarized and given. It is observed that for all the values of ρ^2, PRR values are equal to RR. When $\rho^2 = 10000$, GRR values equal PRR values for lower values of k (from 0.001 to 0.128). It is also observed that from Table 11-13, for lower values of ρ^2, GRR and DRR values yield a lower MSE compared to that of PRR. PRR values equal GRR and DRR for lower values of k (from 0.001 to 0.032).When $\rho^2 = 100$, PRR performs equally well as GRR and DRR at lower values of k (from 0.001 to 0.016). When $\rho^2 = 16, 4$ and 1, GRR and DRR are lesser than PRR. In Table 14-16, the values are equal till $k = 0.032$ when $\rho^2 = 10000$ and 100. For all the methods the MSE values are increasing as ρ^2 decreases. When $k = 1.55$ to 11.55, the MSE are increasing when ρ^2 is 10000, 100, 16, 4 and stabilizing when $\rho^2 = 1$ at $k = 3.55$. From Table 17-19, it is observed that, when ρ^2 is 10000 and 100, GRR and DRR are better than PRR. But PRR outperformed GRR and DRR by a sizeable amount when $\rho^2 = 16, 4, 1$. When $\rho^2 = 16, 4$ and 1, MSE values are less compared to LSE and decreasing for all the values of k in PRR. Table 20-22 shows GRR and DRR are having lesser values than PRR at $\rho^2 = 16, 4$ and 1. The MSE values are less compared to LSE and decreasing for all the values of k in PRR. When $k = 1.55$ to 11.55, the MSE of PRR are increasing when ρ^2 is 10000, 100 and stabilizing when $\rho^2 = 16, 4, 1$. Table 23-25 provides the results as GRR and DRR show lesser values than PRR at $\rho^2 = 16, 4$ and 1. When $\rho^2 = 16, 4$ and 1, the MSE values are less compared to LSE and decreasing for all the values of k in PRR. In Table 26-28 it is observed that PRR is performing well than GRR and DRR at $k = 0.128$, 0.256 and 0.512. when $\rho^2 = 10000$, the MSE values

of PRR are equal to RR till $k = 0.016$. However, when $\rho^2 = 100$, the MSE values are decreasing till $k = 0.256$ and increases from $k = 0.512$. The MSE values of PRR are lower than GRR and DRR for $\rho^2 = 16, 4, 1$.
For high value of $\gamma = 0.99$, PRR values are equal to RR and PRR performs better than GRR and DRR. For 0.9, GRR and DRR perform better than PRR. Therefore, PRR perform better for higher values of γ. PRR produces smaller MSE than the LSE and the MSE values decreases as the sample size and the number of explanatory variables are increased. It is observed that for small sample size and moderate correlation GRR and DRR performs well. Yet, PRR outperforms RR, GRR and DRR when the sample size and variables increases.

5 Application

To compare the performance of the proposed estimators. An example contains 21 days of operations of a plant for the oxidation of ammonia (NH3) to nitric acid (HNO3) The following linear regression method is considered (Brownlee, 1965). The following linear regression method is considered

$$\mathbf{Y} = \beta_0 + \beta_1\mathbf{X}_1 + \beta_2\mathbf{X}_2 + \beta_3\mathbf{X}_3 + \beta_4\mathbf{X}_4 + e_1 \tag{20}$$

where $\mathbf{Y}$ is the percent of ingoing NH3 that is lost by the escaping in the unabsorbed nitric oxide (x10) which is an inverse measure of the yield of the overall efficiency for the plant, $\mathbf{X}_1$ is the operation day, $\mathbf{X}_2$ is the rate of operation of the plant, $\mathbf{X}_3$ is the temperature of the cooling water in the coils of the absorbing tower for the nitric oxide and $\mathbf{X}_4$ is the concentration of HNO3 in the absorbing liquid (coded by minus 50, times 10). The details of the data is given in Brownlee (1965). The correlation matrix of the variables in model (20) is presented in Table 1. The explanatory variables are moderate to highly correlated is observed from the Table 2. The existence of multicollinearity in the data is found by using the device conditional number ($\lambda_1/\lambda_p = 3737.51$).
The MSE of the respective models are obtained by

$$MSE\left(\hat{\beta}\right) = \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i}{\left(\lambda_i + \hat{k}_i\right)^2} + \sum_{i=1}^{p} \frac{\hat{k}_i \alpha_i^2}{\left(\lambda_i + \hat{k}_i\right)^2} \tag{21}$$

The mean square error along with the ridge regression coefficients, OLS are given for different methods are given in the Table 2-4. It is observed that all the regression methods perform better than the least square estimators having smaller mean square error. PRR perform equally with RR and a sensible amount of MSE slightly more than GRR and DRR.

6 Conclusion

This paper concentrates on the comparison of ridge regression methods for the problem of multicollinearity. The problem has been dealt with RR by adding a coefficient

Table 1: Correlation between the variables

	X_1	X_2	X_3	X_4	Y
X_1	1.000	0.696	0.767	0.426	0.810
X_2	0.696	1.000	0.782	0.500	0.920
X_3	0.767	0.782	1.000	0.391	0.876
X_4	0.426	0.500	0.391	1.000	0.400
Y	0.810	0.920	0.876	0.400	1.000

Table 2: GRR, DRR Regression Coefficients and MSE

	OLS	**GRR**	**DRR**
β_1	-0.534	-0.61	-0.61
β_2	0.668	0.655	0.655
β_3	0.64	0.466	0.466
β_4	-0.353	-0.292	-0.291
MSE	9.07	1.77	1.77

k in stabilizing the estimates of regression coefficients and obtaining a smaller MSE. The study with k equals to zero leads the ordinary least square estimators. The computational difficulty in arriving at k and the results thereafter led to introduction of GRR with perturbation of k_i $(i = 1, 2, ..., p)$ constants than a single one proposed in the classical ridge regression. DRR has been identified as an alternative approach with addition of biasing constants k_i $(i = 1, 2, ..., q < p)$ to a selective number of variables. Successively Chandrasekhar et al (2016) have also developed PRR which is based on the cut-off criteria method.

The study gains importance in identifying the best possible method to fit regression in the presence of collinearity. A Monte Carlo simulation study has been conducted to compare the performance of the methods. The methods have been evaluated using the MSE. The MSE is obtained by varying n, p and γ for different levels of ρ^2. In the simulation study, comparison of the RR, GRR, DRR and PRR methods showed that RR, PRR perform equally well when the sample size, explanatory variables and correlation are at a lower level. The MSE values of DRR are moderately less than GRR only when n, p are less with variability in the values of γ. But for the same methods, the values are equal in all the cases of increased values of n, p and differing γ values. PRR shows a considerably lower value of MSE for higher γ values with varying number of observations and explanatory variables. The results have established a small perturbation of a single k as compared to multiple k's. Using a single k gives efficient results in terms of stability and MSE remains considerably less compared to other methods. The study has shown, based on the results obtained by simulation, that PRR is the best method because it overcomes the shortcomings of other ridge regression methods. The

Table 3: RR Regression Coefficients and MSE

k	β_1	β_2	β_3	β_4	MSE
0.000	-0.534	0.668	0.64	-0.353	1.93
0.001	-0.534	0.668	0.64	-0.353	1.93
0.002	-0.534	0.668	0.64	-0.353	1.93
0.004	-0.534	0.668	0.64	-0.353	1.93
0.008	-0.534	0.668	0.64	-0.353	1.93
0.016	-0.534	0.668	0.64	-0.353	1.93
0.032	-0.534	0.668	0.64	-0.353	1.93
0.064	-0.534	0.668	0.64	-0.353	1.93
0.128	-0.534	0.669	0.639	-0.353	1.93
0.256	-0.534	0.669	0.638	-0.353	1.93
0.512	-0.534	0.669	0.637	-0.352	1.93
0.550	-0.5345	0.6688	0.6365	-0.3523	19.287
0.600	-0.5346	0.6689	0.6361	-0.3522	19.284
0.650	-0.5346	0.6689	0.6358	-0.3522	1.928
0.700	-0.5346	0.6689	0.6355	-0.3521	19.276
0.750	-0.5347	0.669	0.6351	-0.352	19.272
0.800	-0.5347	0.669	0.6348	-0.352	19.269
0.850	-0.5347	0.669	0.6345	-0.3519	19.265
0.900	-0.5348	0.6691	0.6342	-0.3519	19.261
0.950	-0.5348	0.6691	0.6338	-0.3518	19.257
1.000	-0.5348	0.6691	0.6335	-0.3517	19.254
1.050	-0.5349	0.6692	0.6332	-0.3517	1.925
1.550	-0.5352	0.6695	0.6299	-0.3511	19.213
2.550	-0.5359	0.6701	0.6236	-0.3499	19.141
3.550	-0.5366	0.6707	0.6174	-0.3487	19.072
4.550	-0.5372	0.6713	0.6114	-0.3476	19.005
5.550	-0.5378	0.6719	0.6056	-0.3465	18.940
6.550	-0.5384	0.6724	0.6000	-0.3454	18.877
7.550	-0.5390	0.6728	0.5944	-0.3443	18.816
8.550	-0.5395	0.6733	0.5891	-0.3432	18.757
9.550	-0.5400	0.6737	0.5838	-0.3422	18.700
10.550	-0.5405	0.6741	0.5787	-0.3411	18.645
11.550	-0.5410	0.6745	0.5738	-0.3401	18.591

Table 4: PRR Regression Coefficients and MSE

k	β_1	β_2	β_3	β_4	MSE
0.000	-0.534	0.668	0.640	-0.353	1.93
0.001	-0.534	0.668	0.640	-0.353	1.93
0.002	-0.534	0.668	0.640	-0.353	1.93
0.004	-0.534	0.668	0.640	-0.353	1.93
0.008	-0.534	0.668	0.640	-0.353	1.93
0.016	-0.534	0.668	0.640	-0.353	1.93
0.032	-0.534	0.668	0.640	-0.353	1.93
0.064	-0.534	0.668	0.640	-0.353	1.93
0.128	-0.534	0.668	0.639	-0.353	1.93
0.256	-0.535	0.669	0.638	-0.353	1.93
0.512	-0.535	0.669	0.636	-0.352	1.93
0.550	-0.5351	10.000	10.000	10.000	19.283
0.600	-0.5352	0.6687	0.6356	-0.3519	19.279
0.650	-0.5353	0.6687	0.6352	-0.3518	19.275
0.700	-0.5354	0.6687	0.6348	-0.3517	19.271
0.750	-0.5355	0.6688	0.6344	-0.3516	19.267
0.800	-0.5355	0.6688	0.6341	-0.3516	19.262
0.850	-0.5356	0.6688	0.6337	-0.3515	19.258
0.900	-0.5357	0.6688	0.6333	-0.3514	19.254
0.950	-0.5358	0.6688	0.6329	-0.3513	19.250
1.000	-0.5359	0.6689	0.6326	-0.3512	19.246
1.050	-0.5360	0.6689	0.6322	-0.3511	19.242
1.550	-0.5369	0.6691	0.6285	-0.3502	19.201
2.550	-0.5386	0.6695	0.6213	-0.3485	19.123
3.550	-0.5403	0.6698	0.6142	-0.3468	19.047
4.550	-0.5420	0.6701	0.6074	-0.3451	18.974
5.550	-0.5437	0.6704	0.6008	-0.3435	18.904
6.750	-0.5453	0.6706	0.5943	-0.3419	18.837
7.550	-0.5470	0.6708	0.5880	-0.3403	18.772
8.950	-0.5485	0.6710	0.5818	-0.3387	18.709
9.550	-0.5501	0.6711	0.5759	-0.3371	18.648
10.550	-0.5517	0.6712	0.5700	-0.3356	18.589
11.550	-0.5532	0.6713	0.5643	-0.3340	18.533

above results can be extended for further evaluation with regard to regression models with compositional explanatory variables since such models are characterized by bad conditionality requiring the use of a biased alternative, in this case, PRR.

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Appendix A

Table 5: RR MSE ($n = 15$, $p = 4$, $Y = 0.7$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0001	0.0059	0.037	0.148	0.5933
0.001	0.0001	0.0059	0.037	0.1479	0.593
0.002	0.0001	0.0059	0.037	0.1478	0.5927
0.004	0.0001	0.0059	0.0369	0.1477	0.592
0.008	0.0001	0.0059	0.0368	0.1473	0.5907
0.016	0.0001	0.0059	0.0367	0.1467	0.588
0.032	0.0001	0.0058	0.0364	0.1454	0.5828
0.064	0.0001	0.0057	0.0357	0.1429	0.5728
0.128	0.0001	0.0056	0.0346	0.1382	0.554
0.256	0.0002	0.0053	0.0326	0.1299	0.5203
0.512	0.0006	0.0052	0.0295	0.1164	0.4651
0.55	0.0006	0.0052	0.0291	0.115	0.4547
0.60	0.0007	0.0052	0.0286	0.1128	0.4458
0.65	0.0009	0.0052	0.0282	0.1108	0.4372
0.70	0.001	0.0052	0.0278	0.1088	0.429
0.75	0.0011	0.0053	0.0274	0.1068	0.4211
0.80	0.0012	0.0053	0.0271	0.105	0.4135
0.85	0.0013	0.0054	0.0267	0.1033	0.4062
0.90	0.0015	0.0054	0.0264	0.1016	0.3992
0.95	0.0016	0.0055	0.0261	0.1	0.3924
1.000	0.0018	0.0056	0.0258	0.0984	0.3859
1.050	0.0019	0.0057	0.0256	0.0969	0.3795
1.550	0.0036	0.0068	0.0238	0.0848	0.3278
2.550	0.0079	0.0104	0.0236	0.0707	0.2581
3.550	0.0133	0.0153	0.026	0.0641	0.2155
4.550	0.0193	0.021	0.03	0.0618	0.1879
5.550	0.026	0.0274	0.0351	0.0622	0.1697
6.750	0.0331	0.0342	0.041	0.0646	0.1576
7.550	0.0405	0.0415	0.0476	0.0683	0.15
8.950	0.0483	0.0492	0.0546	0.073	0.1455
9.550	0.0563	0.057	0.0619	0.0785	0.1434
10.550	0.0644	0.0651	0.0696	0.0845	0.1432
11.550	0.0727	0.0733	0.0774	0.091	0.1443

Table 6: PRR MSE ($n = 15$, $p = 4$, $Y = 0.7$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0001	0.0059	0.037	0.148	0.5933
0.001	0.0001	0.0059	0.037	0.1479	0.593
0.002	0.0001	0.0059	0.037	0.1478	0.5927
0.004	0.0001	0.0059	0.0369	0.1477	0.592
0.008	0.0001	0.0059	0.0368	0.1473	0.5907
0.016	0.0001	0.0059	0.0367	0.1467	0.588
0.032	0.0001	0.0058	0.0364	0.1454	0.5828
0.064	0.0001	0.0057	0.0357	0.1429	0.5728
0.128	0.0001	0.0056	0.0346	0.1382	0.554
0.256	0.0002	0.0053	0.0326	0.1299	0.5203
0.512	0.0006	0.0052	0.0295	0.1164	0.4651
0.550	0.0006	0.0052	0.0291	0.115	0.4547
0.600	0.0007	0.0052	0.0286	0.1128	0.4458
0.650	0.0009	0.0052	0.0282	0.1108	0.4372
0.700	0.001	0.0052	0.0278	0.1088	0.429
0.750	0.0011	0.0053	0.0274	0.1068	0.4211
0.800	0.0012	0.0053	0.0271	0.105	0.4135
0.850	0.0013	0.0054	0.0267	0.1033	0.4062
0.900	0.0015	0.0054	0.0264	0.1016	0.3992
0.950	0.0016	0.0055	0.0261	0.1	0.3924
1.000	0.0018	0.0056	0.0258	0.0984	0.3859
1.050	0.0019	0.0057	0.0256	0.0969	0.3795
1.550	0.0036	0.0068	0.0238	0.0848	0.3278
2.550	0.0079	0.0104	0.0236	0.0707	0.2581
3.550	0.0133	0.0153	0.026	0.0641	0.2155
4.550	0.0193	0.021	0.03	0.0618	0.1879
5.550	0.026	0.0274	0.0351	0.0622	0.1697
6.750	0.0331	0.0342	0.041	0.0646	0.1576
7.550	0.0405	0.0415	0.0476	0.0683	0.15
8.950	0.0483	0.0492	0.0546	0.073	0.1455
9.550	0.0563	0.057	0.0619	0.0785	0.1434
10.550	0.0644	0.0651	0.0696	0.0845	0.1432
11.550	0.0727	0.0733	0.0774	0.091	0.1443

Table 7: GRR, DRR MSE ($n = 15$, $p = 4$, $Y = 0.7$)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0001	0.0001
100	0.0057	0.0054
16	0.0318	0.0285
4	0.1234	0.1083
1	0.4878	0.4254

Table 8: RR MSE ($n = 15$, $p = 4$, $Y = 0.8$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0001	0.0084	0.0526	0.2103	0.8387
0.001	0.0001	0.0084	0.0525	0.2101	0.8379
0.002	0.0001	0.0084	0.0525	0.2099	0.8372
0.004	0.0001	0.0084	0.0524	0.2096	0.8358
0.008	0.0001	0.0083	0.0522	0.2088	0.8329
0.016	0.0001	0.0083	0.0519	0.2074	0.8273
0.032	0.0001	0.0082	0.0512	0.2047	0.8163
0.064	0.0001	0.008	0.0499	0.1994	0.7953
0.128	0.0001	0.0076	0.0475	0.1898	0.7571
0.256	0.0001	0.007	0.0434	0.1735	0.6918
0.512	0.0003	0.0062	0.0373	0.1485	0.5918
0.550	0.0003	0.0061	0.0367	0.1455	0.5816
0.600	0.0004	0.006	0.0357	0.1416	0.5661
0.650	0.0004	0.0059	0.0349	0.138	0.5514
0.700	0.0005	0.0058	0.034	0.1346	0.5374
0.750	0.0005	0.0057	0.0333	0.1313	0.5242
0.800	0.0006	0.0056	0.0325	0.1282	0.5115
0.850	0.0006	0.0056	0.0318	0.1252	0.4995
0.900	0.0007	0.0055	0.0312	0.1224	0.488
0.950	0.0007	0.0055	0.0305	0.1197	0.477
1.000	0.0008	0.0054	0.0299	0.1171	0.4665
1.050	0.0009	0.0054	0.0294	0.1146	0.4564
1.550	0.0017	0.0054	0.0249	0.0946	0.3737
2.550	0.004	0.0067	0.0209	0.0715	0.274
3.550	0.0071	0.0091	0.0201	0.0594	0.2163
4.550	0.0108	0.0124	0.0213	0.0531	0.1797
5.550	0.0151	0.0164	0.0238	0.0502	0.1555
6.750	0.0198	0.0209	0.0272	0.0496	0.1392
7.550	0.0249	0.0259	0.0313	0.0507	0.1282
8.950	0.0304	0.0312	0.036	0.053	0.1209
9.550	0.0362	0.0368	0.0411	0.0562	0.1164
10.550	0.0422	0.0427	0.0466	0.0601	0.114
11.550	0.0484	0.0489	0.0523	0.0645	0.1132

Table 9: PRR MSE ($n = 15$, $p = 4$, $Y = 0.8$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0001	0.0084	0.0526	0.2103	0.8387
0.001	0.0001	0.0084	0.0526	0.2102	0.8383
0.002	0.0001	0.0084	0.0525	0.2101	0.8379
0.004	0.0001	0.0084	0.0525	0.2099	0.8372
0.008	0.0001	0.0084	0.0524	0.2096	0.8358
0.016	0.0001	0.0084	0.0522	0.2089	0.8329
0.032	0.0001	0.0083	0.0519	0.2075	0.8274
0.064	0.0002	0.0083	0.0513	0.2049	0.8167
0.128	0.0005	0.0084	0.0504	0.2001	0.7969
0.256	0.0018	0.0093	0.0494	0.1924	0.7623
0.512	0.006	0.013	0.05	0.1819	0.7081
0.550	0.0068	0.0137	0.0504	0.1808	0.7032
0.600	0.0078	0.0147	0.0509	0.1795	0.6947
0.650	0.009	0.0157	0.0515	0.1783	0.6867
0.700	0.0102	0.0168	0.0521	0.1773	0.6791
0.750	0.0114	0.018	0.0528	0.1765	0.6719
0.800	0.0127	0.0192	0.0536	0.1757	0.6651
0.850	0.014	0.0204	0.0544	0.1751	0.6586
0.900	0.0153	0.0217	0.0553	0.1746	0.6524
0.950	0.0167	0.0231	0.0563	0.1742	0.6466
1.000	0.0182	0.0244	0.0573	0.1739	0.641
1.050	0.0196	0.0258	0.0583	0.1736	0.6358
1.550	0.0355	0.0412	0.0704	0.176	0.5964
2.550	0.0708	0.076	0.1013	0.1932	0.5572
3.550	0.107	0.1117	0.1346	0.2179	0.546
4.550	0.1419	0.1464	0.1676	0.2449	0.5483
5.550	0.1749	0.1791	0.1992	0.2721	0.5576
6.750	0.2057	0.2098	0.2289	0.2986	0.5704
7.550	0.2343	0.2383	0.2568	0.3239	0.5851
8.950	0.2609	0.2648	0.2828	0.3478	0.6006
9.550	0.2856	0.2894	0.307	0.3703	0.6163
10.550	0.3086	0.3123	0.3295	0.3914	0.6318
11.550	0.33	0.3336	0.3505	0.4113	0.647

Table 10: GRR, DRR MSE ($n = 15$, $p = 4$, $Y = 0.8$)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0001	0.0001
100	0.0057	0.0054
16	0.0318	0.0285
4	0.1234	0.1083
1	0.4878	0.4254

Table 11: RR MSE ($n = 15$, $p = 4$, $Y = 0.9$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0002	0.0161	0.1004	0.4016	16.098
0.001	0.0002	0.016	0.1003	0.4009	1.607
0.002	0.0002	0.016	0.1001	0.4002	16.041
0.004	0.0002	0.016	0.0997	0.3988	15.985
0.008	0.0002	0.0159	0.099	0.3961	15.874
0.016	0.0002	0.0156	0.0977	0.3907	15.659
0.032	0.0002	0.0152	0.0951	0.3804	15.248
0.064	0.0002	0.0145	0.0904	0.3618	14.499
0.128	0.0002	0.0132	0.0825	0.33	13.226
0.256	0.0002	0.0114	0.0705	0.2816	11.283
0.512	0.0004	0.009	0.0547	0.2179	0.8727
0.550	0.0004	0.0088	0.0529	0.2112	0.8423
0.600	0.0005	0.0085	0.0508	0.2025	0.8073
0.650	0.0005	0.0082	0.0488	0.1944	0.775
0.700	0.0006	0.0079	0.047	0.187	0.745
0.750	0.0006	0.0077	0.0453	0.18	0.7172
0.800	0.0007	0.0075	0.0437	0.1735	0.6911
0.850	0.0007	0.0073	0.0422	0.1675	0.6668
0.900	0.0008	0.0072	0.0409	0.1618	0.644
0.950	0.0008	0.007	0.0396	0.1565	0.6226
1.000	0.0009	0.0069	0.0384	0.1515	0.6024
1.050	0.0009	0.0067	0.0373	0.1468	0.5834
1.550	0.0016	0.006	0.029	0.111	0.441
2.550	0.0035	0.0062	0.0211	0.0743	0.2873
3.550	0.0058	0.0077	0.0184	0.0566	0.2092
4.550	0.0085	0.01	0.0182	0.0474	0.1638
5.550	0.0117	0.0129	0.0194	0.0427	0.1355
6.750	0.0152	0.0162	0.0215	0.0408	0.117
7.550	0.0191	0.0198	0.0244	0.0406	0.1049
8.950	0.0232	0.0239	0.0278	0.0417	0.0969
9.550	0.0276	0.0282	0.0316	0.0438	0.0919
10.550	0.0322	0.0327	0.0357	0.0466	0.089
11.550	0.0371	0.0375	0.0402	0.0499	0.0878

Table 12: PRR MSE ($n = 15$, $p = 4$, $Y = 0.9$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0002	0.0161	0.1004	0.4016	16.098
0.001	0.0002	0.0161	0.1003	0.4013	16.084
0.002	0.0002	0.016	0.1003	0.4009	1.607
0.004	0.0002	0.016	0.1001	0.4002	16.042
0.008	0.0002	0.016	0.0997	0.3988	15.986
0.016	0.0002	0.0159	0.0991	0.3961	15.877
0.032	0.0003	0.0158	0.0978	0.3909	15.666
0.064	0.0006	0.0157	0.0957	0.3814	15.275
0.128	0.0019	0.0163	0.0925	0.3651	14.588
0.256	0.0061	0.0194	0.0896	0.341	13.496
0.512	0.019	0.0307	0.0924	0.3133	12.004
0.550	0.0213	0.0328	0.0935	0.3112	11.778
0.600	0.0244	0.0357	0.0951	0.3083	11.572
0.650	0.0276	0.0387	0.0969	0.3059	11.382
0.700	0.0308	0.0417	0.0989	0.304	11.207
0.750	0.0342	0.0449	0.101	0.3024	11.044
0.800	0.0376	0.0481	0.1032	0.3011	10.893
0.850	0.041	0.0514	0.1056	0.3002	10.753
0.900	0.0445	0.0547	0.108	0.2995	10.623
0.950	0.0481	0.0581	0.1106	0.2991	10.502
1.000	0.0517	0.0616	0.1132	0.2989	10.389
1.050	0.0553	0.065	0.1159	0.299	10.284
1.550	0.0916	0.0999	0.1459	0.3069	0.958
2.550	0.1632	0.1702	0.2096	0.3466	0.8997
3.550	0.2276	0.2339	0.2697	0.3933	0.892
4.550	0.2841	0.2901	0.3236	0.4387	0.9033
5.550	0.3337	0.3393	0.3713	0.4806	0.9223
6.750	0.3772	0.3827	0.4135	0.5187	0.944
7.550	0.4157	0.4211	0.451	0.5532	0.9663
8.950	0.4499	0.4552	0.4844	0.5843	0.9881
9.550	0.4805	0.4857	0.5144	0.6124	10.089
10.550	0.508	0.5131	0.5414	0.6379	10.286
11.550	0.5329	0.5379	0.5658	0.6612	10.471

Table 13: GRR, DRR MSE ($n = 15$, $p = 4$, $Y = 0.9$)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0002	0.0002
100	0.0101	0.0093
16	0.0264	0.0264
4	0.2307	0.1998
1	0.9227	0.7979

Table 14: RR MSE ($n = 15$, $p = 4$, $Y = 0.99$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0016	0.1561	0.9755	39.039	155.636
0.001	0.0015	0.1533	0.9582	38.343	152.867
0.002	0.0015	0.1506	0.9416	37.678	150.223
0.004	0.0015	0.1457	0.9105	36.431	145.261
0.008	0.0014	0.1368	0.8551	3.421	136.418
0.016	0.0012	0.1223	0.7641	30.565	121.904
0.032	0.001	0.1011	0.6318	25.269	100.798
0.064	0.0008	0.0749	0.468	18.719	7.468
0.128	0.0005	0.0482	0.3014	12.054	48.094
0.256	0.0003	0.0264	0.165	0.66	26.332
0.512	0.0002	0.0123	0.0763	0.3048	12.162
0.550	0.0002	0.0112	0.0696	0.2793	11.164
0.600	0.0002	0.0101	0.0624	0.2501	0.9996
0.650	0.0002	0.0091	0.0563	0.2255	0.901
0.700	0.0002	0.0083	0.051	0.2045	0.817
0.750	0.0003	0.0076	0.0466	0.1864	0.7447
0.800	0.0003	0.007	0.0427	0.1708	0.6821
0.850	0.0003	0.0065	0.0393	0.1572	0.6274
0.900	0.0003	0.006	0.0363	0.1452	0.5794
0.950	0.0003	0.0056	0.0337	0.1346	0.5369
1.000	0.0004	0.0053	0.0314	0.1251	0.4992
1.050	0.0004	0.005	0.0293	0.1167	0.4655
1.550	0.0007	0.0033	0.017	0.0657	0.2613
2.550	0.0018	0.003	0.0094	0.0323	0.1242
3.550	0.0033	0.004	0.0079	0.0219	0.0782
4.550	0.0052	0.0057	0.0084	0.0183	0.058
5.550	0.0074	0.0078	0.0099	0.0176	0.0481
6.750	0.01	0.0103	0.012	0.0183	0.0432
7.550	0.0129	0.0132	0.0146	0.0199	0.0411
8.950	0.016	0.0163	0.0175	0.0222	0.0407
9.550	0.0194	0.0196	0.0207	0.0249	0.0415
10.550	0.023	0.0232	0.0242	0.028	0.043
11.550	0.0268	0.027	0.0279	0.0314	0.0452

Table 15: PRR MSE ($n = 15$, $p = 4$, $Y = 0.99$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0016	0.1561	0.9755	39.039	155.636
0.001	0.0015	0.154	0.9625	38.515	153.551
0.002	0.0016	0.152	0.95	38.012	151.546
0.004	0.0016	0.1483	0.9264	37.059	147.753
0.008	0.0019	0.1418	0.8838	35.339	140.892
0.016	0.0032	0.1317	0.8129	32.455	129.368
0.032	0.0077	0.119	0.7087	28.139	112.024
0.064	0.0223	0.1111	0.5815	22.606	89.516
0.128	0.0641	0.1283	0.4679	16.798	65.095
0.256	0.1687	0.2103	0.4299	12.136	43.376
0.512	0.3894	0.4134	0.5399	0.9914	27.925
0.550	0.4183	0.4434	0.5636	0.9836	26.848
0.600	0.4588	0.4822	0.5936	0.9812	25.528
0.650	0.4983	0.5203	0.6239	0.9833	24.419
0.700	0.5368	0.5576	0.6543	0.9889	23.482
0.750	0.5743	0.5941	0.6847	0.9973	22.686
0.800	0.6108	0.6296	0.7148	10.078	22.007
0.850	0.6463	0.6643	0.7447	10.201	21.426
0.900	0.6809	0.6981	0.7741	10.337	20.927
0.950	0.7145	0.7309	0.803	10.483	20.498
1.000	0.7472	0.763	0.8314	10.637	2.013
1.050	0.7789	0.7942	0.8593	10.797	19.812
1.550	10.544	10.635	11.069	12.535	18.433
2.550	14.341	14.399	1.465	15.467	18.779
3.550	1.683	16.874	1.705	17.602	19.859
4.550	18.588	18.625	18.761	19.177	20.893
5.550	19.896	19.928	20.042	20.378	21.778
6.750	20.908	20.938	21.036	21.322	2.252
7.550	21.715	21.743	21.831	22.083	23.143
8.950	22.374	2.24	22.481	22.709	23.671
9.550	22.922	22.946	23.022	23.232	24.123
10.550	23.385	23.408	2.348	23.677	24.513
11.550	23.782	23.804	23.873	24.059	24.853

Table 16: GRR, DRR MSE ($n = 15$, $p = 4$, $Y = 0.99$)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0008	0.0008
100	0.0762	0.0762
16	0.476	0.476
4	1.9042	1.9042
1	7.5855	7.5864

Table 17: RR MSE ($n = 25$, $p = 10$, $Y = 0.7$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0001	0.0126	0.0785	0.3136	12.552
0.001	0.0001	0.0125	0.0784	0.3134	12.546
0.002	0.0001	0.0125	0.0784	0.3133	12.539
0.004	0.0001	0.0125	0.0783	0.313	12.526
0.008	0.0001	0.0125	0.0781	0.3123	12.501
0.016	0.0001	0.0125	0.0778	0.311	1.245
0.032	0.0001	0.0124	0.0772	0.3085	12.349
0.064	0.0001	0.0122	0.076	0.3037	12.154
0.128	0.0001	0.0118	0.0737	0.2944	11.784
0.256	0.0001	0.0112	0.0695	0.2778	11.119
0.512	0.0002	0.0101	0.0627	0.2503	10.014
0.550	0.0002	0.01	0.0619	0.2465	0.986
0.600	0.0002	0.0098	0.0608	0.242	0.9678
0.650	0.0003	0.0097	0.0597	0.2376	0.9504
0.700	0.0003	0.0095	0.0587	0.2335	0.9336
0.750	0.0003	0.0094	0.0577	0.2294	0.9174
0.800	0.0003	0.0093	0.0567	0.2255	0.9018
0.850	0.0003	0.0091	0.0558	0.2218	0.8867
0.900	0.0004	0.009	0.0549	0.2182	0.8722
0.950	0.0004	0.0089	0.054	0.2147	0.8582
1.000	0.0004	0.0088	0.0532	0.2113	0.8446
1.050	0.0004	0.0087	0.0524	0.2081	0.8315
1.550	0.0007	0.0079	0.0457	0.1806	0.7229
2.550	0.0015	0.0071	0.0369	0.1431	0.5703
3.550	0.0023	0.0069	0.0314	0.1188	0.47
4.550	0.0032	0.0071	0.0278	0.1017	0.3988
5.550	0.0042	0.0075	0.0254	0.0892	0.3455
6.750	0.0052	0.0082	0.0239	0.0797	0.3043
7.550	0.0063	0.0089	0.0228	0.0724	0.2715
8.950	0.0075	0.0098	0.0223	0.0666	0.2449
9.550	0.0087	0.0108	0.0221	0.0621	0.223
10.550	0.0099	0.0119	0.0221	0.0585	0.2047
11.550	0.0112	0.013	0.0224	0.0557	0.1894

Table 18: PRR MSE ($n = 25$, $p = 10$, $Y = 0.7$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0001	0.0126	0.0785	0.3136	12.552
0.001	0.0001	0.0125	0.0784	0.3135	12.546
0.002	0.0001	0.0125	0.0784	0.3133	12.541
0.004	0.0001	0.0125	0.0783	0.313	12.529
0.008	0.0001	0.0125	0.0782	0.3125	12.506
0.016	0.0001	0.0125	0.0779	0.3113	1.246
0.032	0.0001	0.0124	0.0773	0.309	12.369
0.064	0.0001	0.0122	0.0762	0.3046	12.193
0.128	0.0002	0.0119	0.0742	0.2963	11.858
0.256	0.0003	0.0115	0.0705	0.2813	11.253
0.512	0.0009	0.011	0.0647	0.2565	10.242
0.550	0.001	0.011	0.0641	0.2531	1.01
0.600	0.0011	0.011	0.0632	0.249	0.9933
0.650	0.0013	0.011	0.0623	0.2451	0.9772
0.700	0.0015	0.011	0.0615	0.2414	0.9617
0.750	0.0016	0.011	0.0608	0.2378	0.9468
0.800	0.0018	0.011	0.06	0.2343	0.9324
0.850	0.002	0.0111	0.0593	0.231	0.9185
0.900	0.0022	0.0112	0.0587	0.2278	0.9051
0.950	0.0024	0.0112	0.0581	0.2247	0.8922
1.000	0.0026	0.0113	0.0575	0.2218	0.8797
1.050	0.0029	0.0114	0.0569	0.2189	0.8675
1.550	0.0054	0.013	0.0528	0.1953	0.7672
2.550	0.0121	0.0182	0.0504	0.1656	0.6275
3.550	0.0202	0.0254	0.0524	0.1494	0.5382
4.550	0.0294	0.034	0.0573	0.1413	0.4774
5.550	0.0395	0.0436	0.064	0.1382	0.4345
6.750	0.0503	0.054	0.0722	0.1387	0.4038
7.550	0.0617	0.0651	0.0816	0.1419	0.3818
8.950	0.0736	0.0768	0.0918	0.147	0.3663
9.550	0.086	0.089	0.1027	0.1537	0.3557
10.550	0.0987	0.1016	0.1142	0.1616	0.3489
11.550	0.1118	0.1145	0.1262	0.1705	0.3452

Table 19: GRR, DRR MSE ($n = 25$, $p = 10$, $Y = 0.7$)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0002	0.0002
100	0.0089	0.0089
16	0.0432	0.0432
4	0.1584	0.1584
1	0.6153	0.6155

Table 20: RR MSE ($n = 25$, $p = 10$, $Y = 0.8$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0002	0.018	0.1122	0.4491	17.967
0.001	0.0002	0.0179	0.1121	0.4488	17.953
0.002	0.0002	0.0179	0.1121	0.4485	1.794
0.004	0.0002	0.0179	0.1119	0.4478	17.913
0.008	0.0002	0.0179	0.1115	0.4464	17.859
0.016	0.0002	0.0177	0.1109	0.4438	17.752
0.032	0.0002	0.0175	0.1096	0.4386	17.544
0.064	0.0002	0.0171	0.1071	0.4286	17.144
0.128	0.0002	0.0164	0.1025	0.4101	16.407
0.256	0.0002	0.0152	0.0945	0.3783	15.132
0.512	0.0002	0.0132	0.0822	0.3286	13.144
0.550	0.0002	0.013	0.0805	0.3215	12.859
0.600	0.0002	0.0127	0.0786	0.3138	12.549
0.650	0.0003	0.0124	0.0768	0.3064	12.254
0.700	0.0003	0.0122	0.0751	0.2994	11.974
0.750	0.0003	0.0119	0.0734	0.2927	11.706
0.800	0.0003	0.0117	0.0718	0.2864	11.451
0.850	0.0003	0.0114	0.0703	0.2803	11.207
0.900	0.0003	0.0112	0.0689	0.2745	10.973
0.950	0.0004	0.011	0.0675	0.2689	10.749
1.000	0.0004	0.0108	0.0662	0.2635	10.534
1.050	0.0004	0.0106	0.0649	0.2584	10.328
1.550	0.0006	0.0091	0.0544	0.217	0.8653
2.550	0.0011	0.0074	0.0414	0.1633	0.6488
3.550	0.0016	0.0066	0.0336	0.1303	0.5155
4.550	0.0022	0.0063	0.0284	0.108	0.4247
5.550	0.0028	0.0063	0.0249	0.0921	0.3591
6.750	0.0034	0.0064	0.0225	0.0801	0.3096
7.550	0.0041	0.0067	0.0207	0.071	0.271
8.950	0.0048	0.0071	0.0195	0.0638	0.2402
9.550	0.0056	0.0076	0.0186	0.0581	0.2153
10.550	0.0064	0.0082	0.0181	0.0536	0.1947
11.550	0.0072	0.0089	0.0178	0.0499	0.1775

Table 21: PRR MSE ($n = 25$, $p = 10$, $Y = 0.8$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0002	0.018	0.1122	0.4491	17.967
0.001	0.0002	0.0179	0.1121	0.4488	17.955
0.002	0.0002	0.0179	0.1121	0.4485	17.942
0.004	0.0002	0.0179	0.1119	0.4479	17.918
0.008	0.0002	0.0179	0.1116	0.4467	1.787
0.016	0.0002	0.0178	0.111	0.4443	17.774
0.032	0.0002	0.0176	0.1098	0.4396	17.586
0.064	0.0002	0.0172	0.1076	0.4306	17.224
0.128	0.0003	0.0167	0.1035	0.4139	16.555
0.256	0.0006	0.0158	0.0965	0.385	15.391
0.512	0.0016	0.015	0.086	0.34	13.561
0.550	0.0017	0.0149	0.0847	0.3335	13.291
0.600	0.002	0.0149	0.0831	0.3265	13.004
0.650	0.0023	0.0149	0.0817	0.3199	12.731
0.700	0.0026	0.0149	0.0804	0.3136	1.247
0.750	0.0029	0.015	0.0791	0.3077	12.222
0.800	0.0032	0.0151	0.0779	0.302	11.985
0.850	0.0036	0.0152	0.0768	0.2966	11.758
0.900	0.0039	0.0153	0.0758	0.2914	11.541
0.950	0.0043	0.0154	0.0748	0.2865	11.333
1.000	0.0046	0.0156	0.0739	0.2818	11.134
1.050	0.005	0.0158	0.0731	0.2773	10.942
1.550	0.0095	0.0186	0.0672	0.2423	0.9386
2.550	0.0207	0.0277	0.0655	0.2013	0.741
3.550	0.0344	0.0401	0.071	0.1823	0.6242
4.550	0.0498	0.0546	0.081	0.1753	0.55
5.550	0.0668	0.0709	0.0939	0.1758	0.5013
6.750	0.0849	0.0885	0.1089	0.1814	0.4693
7.550	0.104	0.1072	0.1257	0.1906	0.449
8.950	0.124	0.1269	0.1437	0.2026	0.4371
9.550	0.1447	0.1473	0.1629	0.2167	0.4315
10.550	0.166	0.1684	0.1829	0.2324	0.4307
11.550	0.1879	0.1901	0.2036	0.2495	0.4338

Table 22: GRR, DRR MSE ($n = 25$, $p = 10$, $Y = 0.8$)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0002	0.0002
100	0.0108	0.0108
16	0.0572	0.0572
4	0.2202	0.2202
1	0.8718	0.8719

Table 23: RR MSE (n=25, p=10, Y=0.9)

k \ ρ^2	10000	100	16	4	1
0.000	0.0003	0.0344	0.2152	0.8619	34.458
0.001	0.0003	0.0344	0.2149	0.8607	34.406
0.002	0.0003	0.0343	0.2146	0.8594	34.355
0.004	0.0003	0.0342	0.2139	0.8568	34.253
0.008	0.0003	0.034	0.2127	0.8518	34.051
0.016	0.0003	0.0336	0.2102	0.8419	33.657
0.032	0.0003	0.0329	0.2055	0.823	32.901
0.064	0.0003	0.0315	0.1968	0.7881	31.506
0.128	0.0003	0.0291	0.1817	0.7277	29.094
0.256	0.0003	0.0253	0.1582	0.6335	25.327
0.512	0.0003	0.0203	0.1264	0.5058	20.224
0.550	0.0003	0.0197	0.1228	0.491	19.654
0.600	0.0003	0.019	0.1184	0.4731	18.936
0.650	0.0003	0.0184	0.1142	0.4564	18.269
0.700	0.0003	0.0178	0.1103	0.4409	17.646
0.750	0.0003	0.0172	0.1067	0.4263	17.065
0.800	0.0003	0.0167	0.1033	0.4127	16.519
0.850	0.0003	0.0162	0.1001	0.3999	16.006
0.900	0.0003	0.0157	0.0971	0.3879	15.524
0.950	0.0003	0.0152	0.0942	0.3765	15.068
1.000	0.0003	0.0148	0.0916	0.3657	14.637
1.050	0.0003	0.0144	0.089	0.3556	14.229
1.550	0.0004	0.0113	0.0695	0.2763	11.017
2.550	0.0006	0.008	0.0475	0.1881	0.7493
3.550	0.0009	0.0064	0.0356	0.1397	0.5555
4.550	0.0011	0.0054	0.0282	0.1094	0.4338
5.550	0.0014	0.0049	0.0233	0.0889	0.351
6.750	0.0018	0.0046	0.0199	0.0742	0.2914
7.550	0.0021	0.0046	0.0175	0.0633	0.2469
8.950	0.0025	0.0046	0.0157	0.0551	0.2127
9.550	0.003	0.0048	0.0144	0.0486	0.1857
10.550	0.0034	0.005	0.0135	0.0436	0.1641
11.550	0.0039	0.0053	0.0129	0.0395	0.1465

Table 24: PRR MSE (n=25, p=10, Y=0.9)

k \ ρ^2	10000	100	16	4	1
0.000	0.0003	0.0344	0.2152	0.8619	34.458
0.001	0.0003	0.0344	0.2149	0.8608	34.411
0.002	0.0003	0.0343	0.2146	0.8596	34.365
0.004	0.0003	0.0342	0.214	0.8573	34.273
0.008	0.0003	0.0341	0.2129	0.8528	34.092
0.016	0.0003	0.0337	0.2107	0.8439	33.736
0.032	0.0004	0.033	0.2065	0.8268	33.052
0.064	0.0004	0.0318	0.1986	0.7951	31.785
0.128	0.0007	0.0299	0.1851	0.7401	2.958
0.256	0.0016	0.0274	0.1643	0.6538	26.102
0.512	0.0046	0.0256	0.1374	0.5369	21.341
0.550	0.0051	0.0257	0.1347	0.5236	20.809
0.600	0.0059	0.0258	0.1312	0.5073	20.136
0.650	0.0067	0.026	0.128	0.4923	19.509
0.700	0.0075	0.0262	0.1252	0.4784	18.925
0.750	0.0084	0.0266	0.1226	0.4654	18.378
0.800	0.0093	0.027	0.1202	0.4533	17.866
0.850	0.0103	0.0274	0.1181	0.442	17.384
0.900	0.0112	0.0279	0.1162	0.4314	16.931
0.950	0.0122	0.0285	0.1145	0.4215	16.503
1.000	0.0133	0.0291	0.1129	0.4122	16.099
1.050	0.0143	0.0298	0.1115	0.4034	15.717
1.550	0.0263	0.0385	0.1041	0.3375	12.691
2.550	0.0561	0.0649	0.1122	0.2806	0.9524
3.550	0.0919	0.0987	0.1357	0.2678	0.7941
4.550	0.1319	0.1376	0.168	0.2769	0.7099
5.550	0.1753	0.1803	0.2061	0.2989	0.667
6.750	0.2213	0.2257	0.2482	0.3292	0.6495
7.550	0.2693	0.2734	0.2932	0.3653	0.6489
8.950	0.3188	0.3226	0.3404	0.4054	0.6601
9.550	0.3695	0.3731	0.3892	0.4484	0.6797
10.550	0.4209	0.4244	0.4391	0.4937	0.7055
11.550	0.4729	0.4763	0.4898	0.5405	0.736

Table 25: GRR, DRR MSE (n=25, p=10, Y=0.9)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0003	0.0003
100	0.0174	0.0174
16	0.1047	0.1047
4	0.4167	0.4167
1	1.6623	1.6624

Table 26: RR MSE ($n = 25$, $p = 10$, $Y = 0.99$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0033	0.3337	20.849	8.333	332.956
0.001	0.0033	0.3288	20.545	82.116	328.109
0.002	0.0032	0.3241	20.252	80.947	323.438
0.004	0.0032	0.3152	19.697	78.728	314.575
0.008	0.003	0.2991	1.869	74.704	298.506
0.016	0.0027	0.272	16.997	67.941	271.494
0.032	0.0023	0.2314	1.446	57.802	230.992
0.064	0.0018	0.1791	11.197	44.762	178.894
0.128	0.0013	0.1232	0.7698	30.779	123.011
0.256	0.0008	0.0738	0.4608	18.428	73.644
0.512	0.0005	0.0379	0.2367	0.9465	37.821
0.550	0.0004	0.0351	0.2192	0.8748	35.032
0.600	0.0004	0.0318	0.1989	0.7938	31.788
0.650	0.0004	0.0291	0.1815	0.7243	29.004
0.700	0.0004	0.0267	0.1664	0.6642	26.593
0.750	0.0004	0.0246	0.1533	0.6116	24.487
0.800	0.0003	0.0227	0.1417	0.5654	22.635
0.850	0.0003	0.0211	0.1314	0.5245	20.997
0.900	0.0003	0.0196	0.1223	0.4881	19.539
0.950	0.0003	0.0183	0.1142	0.4555	18.235
1.000	0.0003	0.0172	0.1068	0.4263	17.063
1.050	0.0003	0.0161	0.1002	0.3999	16.005
1.550	0.0003	0.0095	0.0585	0.2332	0.9362
2.550	0.0003	0.0047	0.0277	0.1099	0.4409
3.550	0.0004	0.003	0.0165	0.0646	0.2584
4.550	0.0005	0.0022	0.0111	0.0429	0.1709
5.550	0.0007	0.0019	0.0083	0.0309	0.1221
6.750	0.0009	0.0018	0.0066	0.0236	0.0922
7.550	0.0011	0.0018	0.0056	0.0189	0.0726
8.950	0.0014	0.0019	0.0049	0.0157	0.059
9.550	0.0016	0.0021	0.0046	0.0135	0.0493
10.550	0.0019	0.0023	0.0044	0.0119	0.0422
11.550	0.0023	0.0026	0.0044	0.0108	0.0368

Table 27: PRR MSE ($n = 25$, $p = 10$, $Y = 0.99$)

k \ ρ^2	10000	100	16	4	1
0.000	0.0033	0.3337	20.849	8.333	332.956
0.001	0.0033	0.3293	20.575	82.236	328.587
0.002	0.0033	0.3251	20.311	8.118	324.367
0.004	0.0032	0.317	19.809	7.917	316.337
0.008	0.0032	0.3025	18.893	75.507	301.702
0.016	0.0033	0.2779	17.343	69.299	276.897
0.032	0.0043	0.2415	14.998	59.887	239.265
0.064	0.008	0.1964	11.962	47.626	190.157
0.128	0.0202	0.1554	0.8733	34.337	136.672
0.256	0.0562	0.1435	0.6074	22.614	88.734
0.512	0.1567	0.2081	0.4818	1.456	53.542
0.550	0.1741	0.2222	0.48	14.019	50.701
0.600	0.1974	0.2421	0.4819	13.399	47.513
0.650	0.2215	0.2631	0.4873	1.29	44.785
0.700	0.2462	0.2852	0.4957	12.498	42.431
0.750	0.2714	0.3081	0.5065	12.178	40.386
0.800	0.2972	0.3318	0.5194	11.927	38.599
0.850	0.3235	0.3562	0.5341	11.733	3.703
0.900	0.3502	0.3811	0.5504	11.589	35.647
0.950	0.3772	0.4066	0.568	11.487	34.422
1.000	0.4046	0.4326	0.5868	11.422	33.335
1.050	0.4323	0.459	0.6066	11.389	32.368
1.550	0.7204	0.7399	0.8432	1.213	26.739
2.550	12.983	13.109	13.749	1.605	25.007
3.550	18.334	18.431	18.886	20.531	26.838
4.550	2.312	23.201	23.549	24.811	29.575
5.550	27.362	27.433	2.771	28.722	32.482
6.750	31.119	31.184	31.412	32.248	35.309
7.550	34.457	34.518	34.709	35.416	37.967
8.950	37.436	37.493	37.657	38.266	40.431
9.550	40.107	40.161	40.303	40.835	42.701
10.550	42.512	42.564	42.689	4.316	44.789
11.550	44.689	44.738	4.485	45.271	46.709

Table 28: GRR, DRR MSE ($n = 25$, $p = 10$, $Y = 0.99$)

ρ^2	**GRR** KHK	**DRR** KHK
10000	0.0017	0.0017
100	0.1609	0.1609
16	1.0047	1.0047
4	4.0121	4.0121
1	16.0213	16.0213