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Robust estimation of the location and the scale parameters of shifted Gompertz distribution

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In this study, we consider the estimation of the location parameter μ and the scale parameter σ of the shifted Gompertz distribution. We obtain the closed form estimators of these parameters by using the modified maximum likelihood methodology. We also compare the efficiencies of these estimators with the well-known and widely used least squares and maximum likelihood estimators via Monte-Carlo simulation study in terms of bias, mean square error and deficiency criteria. In addition, we evaluate the performances of the proposed estimators when the data set contains outliers or is contaminated. In other words, the robustness properties of the estimators are investigated. A real data set is analyzed to demonstrate the implementation of the estimation methods at the end of the study.

Keywords: Shifted Gompertz distribution, Modified likelihood, Maximum likelihood, Least squares, Monte-Carlo simulation, Robustness.

1 Introduction

The Bass model consists of a simple linear function that describes the process how new products or technologies get adopted in a population, see Bass (1969). The alternatives and the extensions of this model were proposed by several other authors, see for example Mahajan et al. (1990), Meade and Islam (2006).

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Bemmaor (1994) brought a new perspective to the Bass model and showed an alternative derivation of the first purchase density. The Shifted Gompertz (SG) distribution was used to model the propensity of buying the new product or technology, see also Bemmaor and Lee (2002).

In literature, the distributional properties (such as expectation, variance and limit distributions) and the closed-form expression for the quantile function of SG distribution was considered by Jiménez and Jodrá (2009). Torres (2014) obtained the estimators of the scale and the shape parameters of SG distribution by using the least squares (LS), maximum likelihood (ML) and the method of moments (MM) methodologies.

In this paper, our aim is to estimate the location and the scale parameters of the SG distribution. For this purpose, firstly, we obtain the estimators of the unknown parameters by using the LS methodology. Then, we use the well-known and widely used ML methodology. However, the likelihood equations can not be solved explicitly. For solving these equations, we use two different approaches. The first approach is iterative and other one is non-iterative. We use Tiku's modified maximum likelihood (MML) method which is a non-iterative approach, see Tiku (1967, 1968).

The rest of the paper is organized as follows. In section 2, we give some brief description about the SG distribution. The estimation methods for estimating the location and the scale parameters of the SG distribution are examined in Section 3. The performances of the LS, ML and MML estimators are compared by using Monte-Carlo simulation study in the following section. The robustness properties of the estimators are evaluated in Section 5. In Section 6, a real data set is analyzed by using the SG distribution. Comments and conclusions are given in final section.

2 The Shifted Gompertz Distribution

The probability density function (pdf) and the cumulative density function (cdf) of the SG distribution are given as follows

$$f(z) = \frac{1}{\sigma} e^{-\left(z+\eta e^{-z}\right)} \left(1+\eta\left(1-e^{-z}\right)\right),\ z>0,\ \mu\in\mathbb{R},\ \sigma>0,\ \eta>0 \tag{1}$$

and

$$F(z) = e^{-\eta e^{-z}} \left(1-e^{-z}\right),\ z>0,\ \eta>0, \tag{2}$$

respectively. Here, $z=(x-\mu)/\sigma$, μ is the location parameter, σ is the scale parameter and η is the shape parameter. If the random variable X has the SG distribution with μ, σ and η parameters, then it is denoted by $X \sim$SG(μ,σ,η). It should be noted that the shape parameter η is assumed to be known throughout the estimation process.

The SG distribution reduces to the well-known exponential distribution when $\eta \to 0$. For better understanding the shape of the SG distribution, see Figure 1.

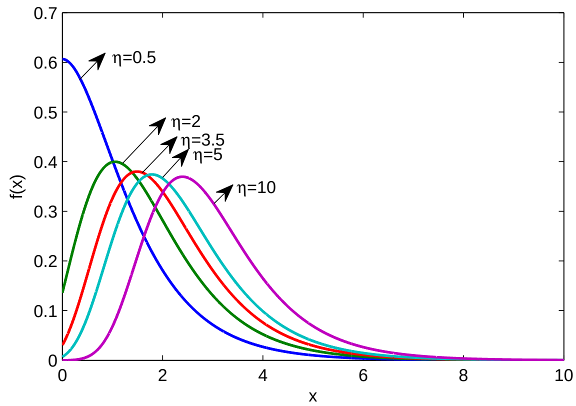

Figure 1: Plots of the SG distribution for different values of the shape parameter η.

Jiménez and Jodrá (2009) provided the explicit expressions for the expectation and the variance of SG distribution. However, the third and the fourth moments of the SG distribution cannot be obtained analytically because of the computational difficulties. Therefore, they tabulated the skewness ($\sqrt{\beta_1}$) and the kurtosis (β_2) values of SG distribution under different values of the shape parameter η numerically. For some selected values of η, the skewness and the kurtosis values of SG distribution are given in the following Table 1.

Table 1: The skewness and the kurtosis values for different values of η.

η	0.01	0.5	1	5	10
$\sqrt{\beta_1}$	1.99	1.71	1.54	1.19	1.15
β_2	5.96	4.45	3.67	2.51	2.43

It is clear from Table 1 and Figure 1 that the SG distribution is positively skewed. In addition, it is long tailed when $\eta \to 0$ and short tailed when $\eta \to \infty$, see Jiménez and Jodrá (2009) for more detailed information about the shape of the SG distribution.

3 Parameter Estimation

In this section, we describe the methodologies used for estimating the location and the scale parameters of the SG distribution.

3.1 Least squares estimators

The LS estimators of the location parameter μ and the scale parameter σ of the SG distribution are obtained by minimizing the following equation

$$A\left(\mu,\sigma\right)=\sum_{i=1}^{n}\left(F\left(z_{(i)}\right)-\frac{i}{n+1}\right)^{2} \tag{3}$$

with respect to these parameters, see Swain et al. (1988). Here, $z_{(i)}=\left(x_{(i)}-\mu\right)/\sigma$ are the standardized ordered observations. $F\left(z_{(i)}\right)$, $(i=1,2,\ldots,n)$ is the cdf for the ith ordered observation and $\frac{i}{n+1}$ is the expected value of $F\left(z_{(i)}\right)$. By incorporating the distribution function of the SG distribution into the equation (3), $A\left(\mu,\sigma\right)$ is obtained as given below

$$A\left(\mu,\sigma\right)=\sum_{i=1}^{n}\left(e^{-\eta\, e^{-z_{(i)}}}\left(1-e^{-z_{(i)}}\right)-u_{i}\right)^{2}, \tag{4}$$

where $u_i=\frac{i}{n+1}$. The derivatives of the $A\left(\mu,\sigma\right)$ with respect to the unknown parameters μ and σ are obtained as follows

$$\frac{\partial A\left(\mu,\sigma\right)}{\partial\mu}=\sum_{i=1}^{n}e^{-\left(z_{(i)}+\eta\, e^{-z_{(i)}}\right)}\left(1+\eta\left(1-e^{-z_{(i)}}\right)\right)\left(e^{-\eta\, e^{-z_{(i)}}}\left(1-e^{-z_{(i)}}\right)-u_{i}\right)=0 \tag{5}$$

$$\frac{\partial A\left(\mu,\sigma\right)}{\partial\sigma}=\sum_{i=1}^{n}z_{(i)}e^{-\left(z_{(i)}+\eta\, e^{-z_{(i)}}\right)}\left(1+\eta\left(1-e^{-z_{(i)}}\right)\right)\left(e^{-\eta\, e^{-z_{(i)}}}\left(1-e^{-z_{(i)}}\right)-u_{i}\right)=0, \tag{6}$$

respectively. The solutions of these equations cannot be obtained explicitly. Therefore, we resort to numerical methods to solve them.

3.2 Maximum Likelihood Estimators

Let $X_1,X_2,\ldots,X_n$ be a random sample from the SG distribution. The log-likelihood function $(\ln L)$ is obtained as follows

$$\ln L\left(\mu,\sigma\right)=-n\ln\sigma-\sum_{i=1}^{n}\left(z_i+\eta e^{-z_i}\right)+\sum_{i=1}^{n}\ln\left(1+\eta\left(1-e^{-z_i}\right)\right), \tag{7}$$

where, $z_i=\left(x_i-\mu\right)/\sigma$. Likelihood equations are obtained by taking the derivatives of the $\ln L$ function with respect to the parameters of interest and equating them to zero. The likelihood equations for parameters μ and σ are given as shown below

$$\frac{\partial\ln L\left(\mu,\sigma\right)}{\partial\mu}=n-\eta\sum_{i=1}^{n}g_1\left(z_i\right)-\eta\sum_{i=1}^{n}g_2\left(z_i\right)=0 \tag{8}$$

$$\frac{\partial\ln L\left(\mu,\sigma\right)}{\partial\sigma}=n-\sum_{i=1}^{n}z_i+\eta\sum_{i=1}^{n}z_ig_1\left(z_i\right)+\eta\sum_{i=1}^{n}z_ig_2\left(z_i\right)=0, \tag{9}$$

respectively. The simultaneous solution of the equations (8) and (9) gives the ML estimates of the μ and σ, respectively. However is clear that these equations do not have explicit solutions because of the nonlinear functions $g_1(z_i) = e^{-z_i}$ and $g_2(z_i) = \frac{e^{-z_i}}{1+\eta\left(1-e^{-z_i}\right)}$ in (8) and (9). Similar to the LS methodology, we resort to iterative methods to solve them, see Luceño (2008).

3.3 Modified Maximum Likelihood Estimators

In the previous subsection, we do not obtain the ML estimators of the location and the scale parameters in explicit form. Therefore, we use MML methodology originated by Tiku (1967, 1968) for deriving the estimators of the unknown parameters in non-iterative ways. This methodology is based on the idea of linearization of the nonlinear terms in the likelihood equations.

Let $Z_{(1)} < Z_{(2)} < \ldots < Z_{(n)}$ be the order statistics obtained by arranging $Z_{(i)}$ $(i = 1, 2, \ldots, n)$ in ascending order of magnitude and $z_{(1)} < z_{(2)} < \ldots < z_{(n)}$ be observed values of $Z_{(1)} < Z_{(2)} < \ldots < Z_{(n)}$. Then, the likelihood equations in (8) and (9) are written in terms of the order statistics, since complete sums are invariant to ordering, i.e., $\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} z_{(i)}$. The nonlinear functions $g_1\left(z_{(i)}\right)$ and $g_2\left(z_{(i)}\right)$ are linearized by using the first two terms of the Taylor series expansion around the expected values of the order statistics, i.e., $t_{(i)} = E\left(z_{(i)}\right)$. Then, we get

$$g_1\left(z_{(i)}\right) \cong \alpha_{1i} - \beta_{1i} z_{(i)} \text{ and } g_2\left(z_{(i)}\right) \cong \alpha_{2i} - \beta_{2i} z_{(i)}, \tag{10}$$

where

$\alpha_{1i} = e^{-t_{(i)}} + t_{(i)}\beta_{1i}$, $\beta_{1i} = e^{-t_{(i)}}$, $\alpha_{2i} = \frac{e^{-t_{(i)}}}{1+\eta\left(1-e^{-t_{(i)}}\right)} + t_{(i)}\beta_{2i}$, $\beta_{2i} = \frac{(1+\eta)e^{-t_{(i)}}}{\left(1+\eta\left(1-e^{-t_{(i)}}\right)\right)^2}$ and $t_{(i)} = -\log\left(1 - \frac{1}{\eta} W_0\left(\eta e^{\eta}\left(\frac{i}{n+1}\right)\right)\right)$.

Here, W_0 is the solution of the equation $W(t)\, e^{W(t)} = t$ and is the principal branch of the Lambert W function. For more detailed information about the Lambert W function, see Barry et al. (2000), Jiménez and Jodrá (2009) and Torres (2014). By incorporating the linearized functions in equation (10) into the likelihood equations (8) and (9), we obtain the following modified likelihood equations

$$\frac{\partial \ln L^*(\mu,\sigma)}{\partial \mu} = n - \eta \sum_{i=1}^{n} \left(\alpha_{1i} - \beta_{1i} z_{(i)}\right) - \eta \sum_{i=1}^{n} \left(\alpha_{2i} - \beta_{2i} z_{(i)}\right) = 0 \tag{11}$$

$$\frac{\partial \ln L^*(\mu,\sigma)}{\partial \sigma} = n - \sum_{i=1}^{n} z_{(i)} + \eta \sum_{i=1}^{n} z_{(i)} \left(\alpha_{1i} - \beta_{1i} z_{(i)}\right) - \eta \sum_{i=1}^{n} z_{(i)} \left(\alpha_{2i} - \beta_{2i} z_{(i)}\right) = 0. \tag{12}$$

The solutions of these equations are the MML estimators of the parameters μ and σ, respectively They are given by

$$\widehat{\mu} = K + D\widehat{\sigma} \text{ and } \widehat{\sigma} = \frac{-B + \sqrt{B^2 + 4nC}}{2\sqrt{n(n-1)}}. \tag{13}$$

Here,

$K = \frac{1}{m}\sum_{i=1}^{n}\delta_i x_{(i)}$, $m = \sum_{i=1}^{n}\delta_i$, $\delta_i = \eta\left(\beta_{1i}+\beta_{2i}\right)$, $D = \frac{\Delta}{m}$, $\Delta = \sum_{i=1}^{n}\Delta_i$,

$\Delta_i = 1-\eta\left(\alpha_{1i}+\alpha_{2i}\right)$, $B = \sum_{i=1}^{n}\Delta_i\left(K - x_{(i)}\right)$ and $C = \sum_{i=1}^{n}\delta_i x_{(i)}^2 - mK^2$.

It is obvious that the MML estimators are easy to compute, since they are functions of the sample observations and have closed from expressions. They have the following properties:

(i) They are non-iterative and give us explicit estimators of the unknown parameters.

(ii) Asymptotically, they are fully efficient. They have high efficiency even for small sample sizes, see Smith et al. (1973), Şenoğlu and Tiku (2001).

(iii) They are asymptotically equivalent to the ML estimators, see Tiku and Suresh (1992), Vaughan (1992).

(iv) The MML methodology can be applied to all distributions belonging to the unimodal location-scale family, see Tiku et al. (1986) and Tiku and Akkaya (2004).

4 Simulation Study

In this section, we compare the performances of the LS, the ML and the MML estimators of the location parameter μ and the scale parameters σ of the SG distribution via Monte Carlo simulation study. Bias and the mean square error (MSE) criteria are used in the comparisons. We also use the deficiency (Def) criterion for the natural measure of the joint efficiencies of $\widehat{\mu}$ and $\widehat{\sigma}$. It is defined as shown below

$$Def\left(\widehat{\mu},\widehat{\sigma}\right) = MSE\left(\widehat{\mu}\right) + MSE\left(\widehat{\sigma}\right), \tag{14}$$

see Tiku and Akkaya (2004). The MSEs of $\widehat{\mu}$ and $\widehat{\sigma}$ are calculated from the following equalities

$$MSE\left(\widehat{\mu}\right) = E\left(\widehat{\mu}-\mu\right)^2 \text{ and } MSE\left(\widehat{\sigma}\right) = E\left(\widehat{\sigma}-\sigma\right)^2, \tag{15}$$

respectively.

We use the following equality known as inverse transformation method

$$z = F_z^{-1}\left(u\right) = -\log\left(1-\frac{1}{\eta}W_0\left(\eta e^{\eta}u\right)\right),\ 0<u<1 \tag{16}$$

to generate the random numbers from the SG distribution.

The performances of the ML, the MML and the LS estimators of the parameters μ and σ are compared for the different sample sizes and the different shape parameters. We used the following settings n = 20, 30, 40, 50, 100, 500, 1000 and η = 2, 3.5, 5, 10, throughout the simulation study.

The location parameter μ and the scale parameter σ are taken to be 0 and 1 without loss of generality. All the simulations are conducted for $[\![100,000/n]\!]$ Monte-Carlo runs by using Matlab 2012b. Here, $[\![\cdot]\!]$ denotes the integer value function. It should be noted that we use fminsearch function in the optimization toolbox of Matlab 2012b to obtain the ML and LS estimates of the parameters. The results are reported in Tables 2 and 3.

It is clear from Table 2 that the LS estimators of the parameters μ and σ have smaller bias than the corresponding ML and the MML estimators when $\eta = 2$ for sample sizes $n \leq 50$. For the large sample sizes ($n \geq 100$) all estimators have negligible bias. Similarly, the LS estimator of μ performs better than the others in terms of MSEs in case of $n \leq 50$. The ML and the MML estimators of σ outperform the LS estimator when $n \leq 100$. However, when the sample sizes get larger, the LS estimator of σ is the most efficient estimator among the others. The LS estimator shows the strongest performance with the lowest deficiency for $n \geq 100$.

In case of $\eta = 3.5$, the ML estimator of μ has the smallest bias among the others when $n \leq 50$. On the other hand, when $n \geq 100$ the LS estimator of the location parameter has smaller bias than the others. In terms of MSE of the estimators of μ, the ML estimator shows superior performance for all sample sizes. It is followed by the MML estimator.

However, when $n \geq 500$ the MSEs of the all estimators are more or less the same. For the estimator of σ, the MML estimator has the smallest bias when $n \leq 100$. The LS estimator has smaller bias than the ML and the MML estimators when $n \geq 500$. The efficiencies of the ML and the MML estimators of σ are more or less the same and they show better performance than the corresponding LS estimator with respect to the MSE criterion. Also, the performances of all the estimators are close to each other when the sample size increases. The ML estimators demonstrate the strongest performance with the lowest deficiency for all sample sizes.

The ML and the MML estimators of μ have smaller bias than the corresponding LS estimator when $\eta = 5$ and $\eta = 10$. In addition to this, the ML estimator of μ demonstrates the strongest performance among the others. It should be noted that the performances of the ML and the MML estimators are close to each other when the sample size n increases as expected. For $\widehat{\sigma}$, the MML estimator exhibits less bias for all the sample sizes. The performances of the ML and the MML estimators are more or less the same with respect to MSE criterion. The LS estimators of μ and σ show the worst performances among the others. It is clear to say that the ML and the MML estimators show the strongest performances with the lowest deficiencies.

It should be noted that the performances of the ML, the MML and the LS estimators are also compared for $\eta = 0.5$. However, the MML estimators cannot be calculated for this case, since the mode of the SG distribution does not exist, see Jiménez and Jodrá (2009) and the property (iv) in subsection 3.3. We therefore did not reproduce the simulation results corresponding to $\eta = 0.5$ for the sake of completeness.

It should be stated that the estimators of μ which are greater than the smallest order statistics $x_{(1)}$, called as impermissible estimators, are replaced by $x_{(1)} - 10^{-4}$ during the simulation study, see Dubey (1967) and Kantar and Senoglu (2008).

Table 2: Bias, MSE and Def values for the LS, the ML and the MML estimators of μ and σ of the SG distribution based on $\eta = 2$ and $\eta = 3.5$.

				$\eta = 2$		
n		$Bias(\hat{\mu})$	$MSE(\hat{\mu})$	$Bias(\hat{\sigma})$	$MSE(\hat{\sigma})$	$Def(\hat{\mu}, \hat{\sigma})$
20	LS	0.0711	0.0464	-0.0580	0.0600	0.1064
	ML	-0.1690	0.0579	0.0906	0.0352	0.0931
	MML	-0.1817	0.0622	0.0741	0.0344	0.0966
30	LS	0.0525	0.0260	-0.0455	0.0386	0.0647
	ML	-0.1349	0.0358	0.0810	0.0238	0.0596
	MML	-0.1431	0.0379	0.0733	0.0235	0.0614
40	LS	0.0424	0.0161	-0.0384	0.0285	0.0446
	ML	-0.1157	0.0250	0.0756	0.0193	0.0444
	MML	-0.1212	0.0262	0.0722	0.0195	0.0457
50	LS	0.0371	0.0113	-0.0332	0.0221	0.0334
	ML	-0.0981	0.0175	0.0749	0.0157	0.0332
	MML	-0.1021	0.0182	0.0739	0.0161	0.0343
100	LS	0.0212	0.0037	-0.0241	0.0109	0.0146
	ML	-0.0593	0.0061	0.0693	0.0105	0.0166
	MML	-0.0600	0.0062	0.0728	0.0113	0.0174
500	LS	0.0063	0.0002	-0.0101	0.0022	0.0025
	ML	-0.0140	0.0003	0.0629	0.0050	0.0054
	MML	-0.0140	0.0003	0.0689	0.0058	0.0062
1000	LS	0.0037	0.0001	-0.0048	0.0013	0.0014
	ML	-0.0068	0.0001	0.0599	0.0041	0.0042
	MML	-0.0068	0.0001	0.0661	0.0049	0.0050
			$\eta = 3.5$			
20	LS	0.1399	0.1215	-0.0869	0.0685	0.1900
	ML	-0.1184	0.0782	0.0538	0.0310	0.1092
	MML	-0.1205	0.0805	0.0295	0.0311	0.1115
30	LS	0.1108	0.0752	-0.0730	0.0439	0.1190
	ML	-0.0924	0.0515	0.0415	0.0197	0.0712
	MML	-0.0962	0.0532	0.0273	0.0198	0.0729
40	LS	0.0922	0.0503	-0.0635	0.0318	0.0821
	ML	-0.0804	0.0375	0.0357	0.0156	0.0530
	MML	-0.0844	0.0388	0.0258	0.0156	0.0544
50	LS	0.0812	0.0376	-0.0556	0.0248	0.0624
	ML	-0.0710	0.0287	0.0336	0.0117	0.0404
	MML	-0.0753	0.0298	0.0266	0.0118	0.0416
100	LS	0.0508	0.0151	-0.0404	0.0122	0.0273
	ML	-0.0549	0.0132	0.0265	0.0066	0.0198
	MML	-0.0579	0.0136	0.0242	0.0067	0.0203
500	LS	0.0186	0.0021	-0.0181	0.0023	0.0044
	ML	-0.0277	0.0022	0.0182	0.0014	0.0037
	MML	-0.0287	0.0023	0.0184	0.0015	0.0038
1000	LS	0.0136	0.0008	-0.0126	0.0011	0.0019
	ML	-0.0195	0.0008	0.0145	0.0006	0.0016
	MML	-0.0200	0.0008	0.0149	0.0007	0.0016

Table 3: Bias, MSE and Def values for the LS, the ML and the MML estimators of μ and σ of the SG distribution based on $\eta = 5$ and $\eta = 10$.

			$\eta = 5$			
n		$Bias(\hat{\mu})$	$MSE(\hat{\mu})$	$Bias(\hat{\sigma})$	$MSE(\hat{\sigma})$	$Def(\hat{\mu}, \hat{\sigma})$
20	LS	0.1794	0.1959	-0.0979	0.0729	0.2688
	ML	-0.1161	0.1100	0.0489	0.0309	0.1409
	MML	-0.1068	0.1112	0.0230	0.0312	0.1423
30	LS	0.1519	0.1287	-0.0853	0.0507	0.1794
	ML	-0.0746	0.0694	0.0304	0.0213	0.0907
	MML	-0.0714	0.0707	0.0141	0.0216	0.0923
40	LS	0.1370	0.0919	-0.0787	0.0375	0.1294
	ML	-0.0598	0.0497	0.0263	0.0146	0.0642
	MML	-0.0604	0.0510	0.0153	0.0147	0.0657
50	LS	0.1206	0.0728	-0.0699	0.0300	0.1028
	ML	-0.0453	0.0396	0.0186	0.0120	0.0516
	MML	-0.0463	0.0406	0.0102	0.0122	0.0528
100	LS	0.0861	0.0341	-0.0552	0.0144	0.0484
	ML	-0.0285	0.0187	0.0112	0.0055	0.0242
	MML	-0.0312	0.0192	0.0084	0.0057	0.0249
500	LS	0.0350	0.0063	-0.0207	0.0030	0.0093
	ML	-0.0100	0.0034	0.0070	0.0012	0.0046
	MML	-0.0114	0.0035	0.0069	0.0012	0.0047
1000	LS	0.0232	0.0030	-0.0169	0.0015	0.0045
	ML	-0.0078	0.0022	0.0046	0.0005	0.0027
	MML	-0.0086	0.0022	0.0047	0.0005	0.0028
			$\eta = 10$			
20	LS	0.2813	0.4174	-0.1140	0.0834	0.5008
	ML	-0.1108	0.1769	0.0382	0.0314	0.2083
	MML	-0.0770	0.1758	0.0097	0.0320	0.2078
30	LS	0.2339	0.2726	-0.0977	0.0566	0.3292
	ML	-0.0742	0.1141	0.0256	0.0204	0.1345
	MML	-0.0547	0.1139	0.0074	0.0206	0.1346
40	LS	0.2085	0.2084	-0.0898	0.0438	0.2521
	ML	-0.0557	0.0867	0.0190	0.0146	0.1013
	MML	-0.0433	0.0871	0.0061	0.0148	0.1019
50	LS	0.1812	0.1587	-0.0799	0.0350	0.1937
	ML	-0.0484	0.0693	0.0163	0.0124	0.0817
	MML	-0.0388	0.0693	0.0060	0.0125	0.0817
100	LS	0.1269	0.0768	-0.0577	0.0179	0.0947
	ML	-0.0263	0.0334	0.0100	0.0060	0.0394
	MML	-0.0232	0.0336	0.0054	0.0060	0.0396
500	LS	0.0461	0.0119	-0.0249	0.0031	0.0150
	ML	-0.0140	0.0067	0.0044	0.0013	0.0079
	MML	-0.0142	0.0067	0.0038	0.0013	0.0080
1000	LS	0.0362	0.0106	-0.0199	0.0029	0.0135
	ML	-0.0024	0.0039	-0.0005	0.0007	0.0046
	MML	-0.0031	0.0039	-0.0006	0.0007	0.0046

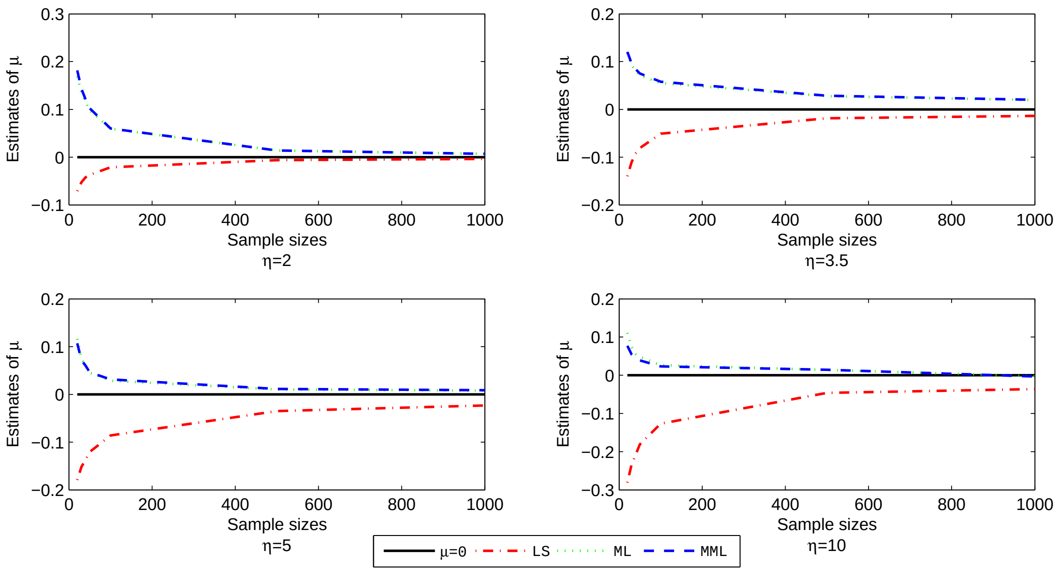

Figure 2: Plots of the LS, the ML and the MML estimates of μ for $n = 20, 30, 40, 50, 100, 500$ and 1000 and $\eta = 2, 3.5, 5$ and 10.

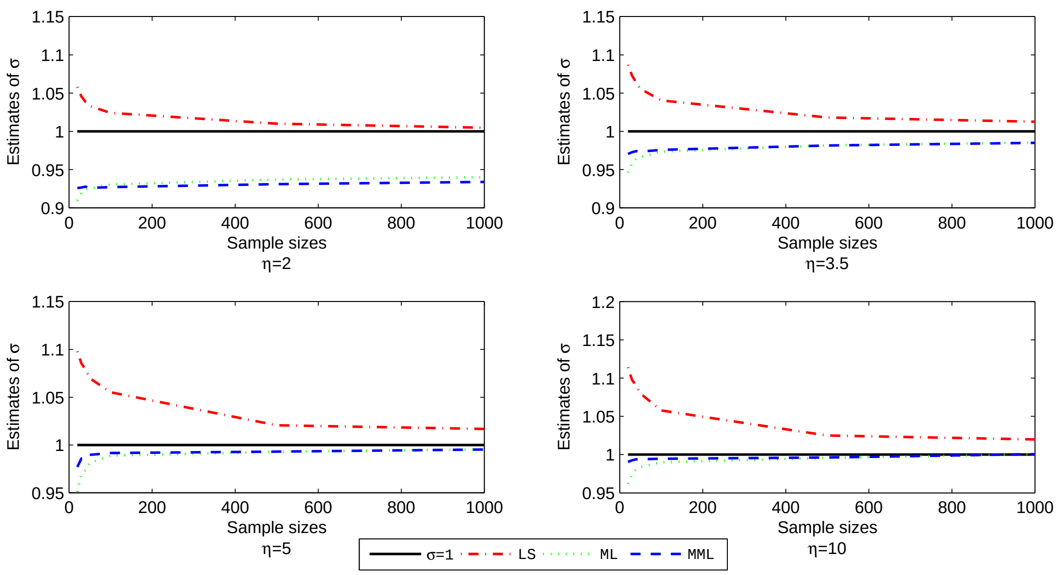

Figure 3: Plots of the LS, the ML and the MML estimates of σ for $n = 20, 30, 40, 50, 100, 500$ and 1000 and $\eta = 2, 3.5, 5$ and 10.

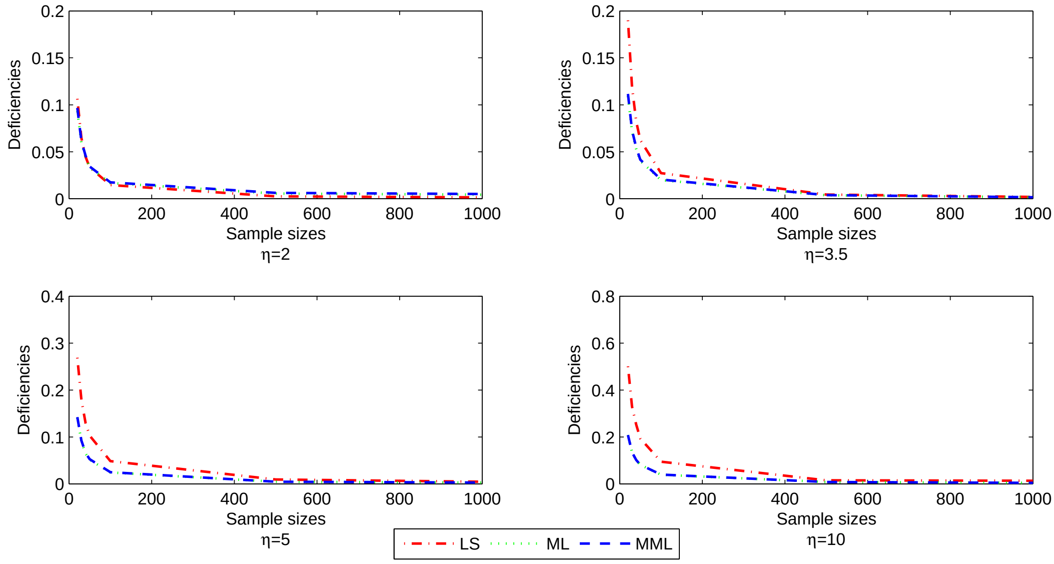

Figure 4: Plots of the deficiencies of the LS, the ML and the MML estimators for $n = 20, 30, 40, 50, 100, 500$ and 1000 and $\eta = 2, 3.5, 5$ and 10.

For better understanding the simulation results, we draw the plots of the LS, the ML and the MML estimates of μ and σ according to the different sample sizes when the shape parameter is equal to 2, 3.5, 5 and 10, see Figures 2 and 3.

In Figure 4, the plots of the deficiencies of the LS, the ML and the MML estimators are also given. It is obvious from Figures 2 and 3 that the bias of the ML and the MML estimators of μ and σ are negligible when the sample size n and the shape parameter η increase. Similarly, from Figure 4, the ML and the MML estimators yield the smallest deficiency when the sample size n and the shape parameter η increase.

5 Robustness Properties

In this section, we compare the performances of the LS, the ML and the MML estimators of the parameters when the data set contains outliers, or is contaminated. In other words, the robustness of the estimators to the plausible deviations from the assumed model is investigated.We assume that the underlying distribution is $SG\,(\mu,\sigma,\eta = 3.5)$, this is called as true model, and consider the following alternative models:

(i) Model I: Misspecified model: $SG\,(\mu,\sigma,\eta = 2)$,

(ii) Model II : Contaminated model: $0.90SG\,(\mu,\sigma,\eta) + 0.10U\,(0, 4.5)$,

(iii) Model III : Mixture model: $0.90SG\,(\mu,\sigma,\eta) + 0.10SG\,(\mu,2\sigma,\eta)$,

(iv) Model IV : Dixon Outlier model: $(n - r)\,SG\,(\mu,\sigma,\eta) + rSG\,(\mu,2\sigma,\eta)$,

where $r = [\![0.5 + 0.1n]\!]$.

For an illustrative purpose, the sample size n is taken to be equal 100. The simulated bias, the MSE and the Def values of the proposed estimators of μ and σ are calculated by using these alternative models. As in Section 4, we take μ and σ as 0 and 1, respectively. The simulation results are reported in Table 4.

Table 4: Bias, MSE and Def values of the estimators of μ and σ under alternative models.

	$Bias(\hat{\mu})$	$MSE(\hat{\mu})$	$Bias(\hat{\sigma})$	$MSE(\hat{\sigma})$	$Def(\hat{\mu},\hat{\sigma})$
			True model		
LS	0.0596	0.0160	-0.0462	0.0124	0.0284
ML	-0.0527	0.0133	0.0200	0.0055	0.0188
MML	-0.0603	0.0150	0.0176	0.0056	0.0206
			Model I		
LS	0.1753	0.0367	0.1302	0.0235	0.0602
ML	0.1595	0.0338	0.0984	0.0145	0.0483
MML	0.1269	0.0235	0.0838	0.0119	0.0355
			Model II		
LS	0.0917	0.0214	-0.0669	0.0149	0.0363
ML	-0.0193	0.0112	0.0007	0.0049	0.0162
MML	-0.0230	0.0115	-0.0017	0.0051	0.0166
			Model III		
LS	0.0652	0.0181	-0.0994	0.0216	0.0397
ML	0.0164	0.0143	-0.0999	0.0193	0.0336
MML	0.0171	0.0147	-0.1073	0.0214	0.0362
			Model IV		
LS	0.0653	0.0193	-0.1013	0.0223	0.0417
ML	0.0155	0.0138	-0.1013	0.0177	0.0315
MML	0.0155	0.0142	-0.1086	0.0197	0.0338

It is clear from Table 4 that the LS estimators of the parameters are the most sensitive to the deviations from the true model. However, the ML and the MML estimators are robust to these alternative models. It is obvious that the ML estimators are the most preferable among the others in terms of MSE and Def criteria.

6 Real data application

In this section, we analyze a real data set taken from the literature to demonstrate the implementation of the estimation methods given in Section 3. For this purpose, we use the strength data which is originally reported by Bader and Priest (1982). The data set is about the breaking strengths of the single carbon fibers of different lengths (i.e., 10, 20, 50). In this study, we only focus on single fibers tested under tension at gauge length of 10mm. The data contains 64 observations and are given in Table 5.

Table 5: The strength data set.

1.901	2.397	2.532	2.738	2.996	3.243	3.435	3.871
2.132	2.445	2.575	2.740	3.030	3.264	3.493	3.886
2.203	2.454	2.614	2.856	3.125	3.272	3.501	3.971
2.228	2.454	2.616	2.917	3.139	3.294	3.537	4.024
2.257	2.474	2.618	2.928	3.145	3.332	3.554	4.027
2.350	2.518	2.624	2.937	3.220	3.346	3.562	4.422
2.361	2.522	2.659	2.937	3.223	3.377	3.628	4.395
2.396	2.525	2.675	2.977	3.235	3.408	3.852	5.020

This data set is modeled by using several statistical distributions, such as Burr X, Weibull and generalized logistic, see Raqab and Kundu (2005), Kundu and Gupta (2006) and Gupta and Kundu (2010). Different than these studies, we use the SG distribution for modeling the strength data.Before analyzing the data set, we identify the shape parameter η by using the profile likelihood method, see for example Islam and Tiku (2004) and Acitas et al. (2013)

The steps of this method are given as follows:

Step 1. For the given value of η, calculate $\widehat{\mu}$ and $\widehat{\sigma}$.

Step 2. Calculate the log-likelihood value by incorporating $\widehat{\mu}$ and $\widehat{\sigma}$ into (7).

Step 3. Repeat step 1 and step 2 for a serious values of η.

Step 4. Find η value maximizing the log-likelihood function among the others and choose it as plausible value of the shape parameter.

Following these steps, we identify the shape parameter η as 3.4. In addition to profile likelihood method for different values of η, we draw the Q-Q plots of the observations. Then, we realized that the Q-Q plots of the SG distribution with $\eta = 3.4$ do not deviate too much from the straight line for the strength data, see Figure 5.

It is clear that both the profile likelihood methodology and the Q-Q plot technique are in agreement for identifying the shape parameter of the SG distribution. Based on the shape parameter $\eta = 3.4$, the ML, the MML and the LS estimates of the parameters μ and σ are obtained, see Table 6. Also, the bootstrap standard errors (SEs) of the corresponding estimators are reported in Table 6.

It is clear from Table 6 that the ML and the MML estimates of μ and σ are very close to each other. According to the bootstrap SEs of the LS, the ML and the MML estimates, the MML estimates of the parameters μ and σ are the most reliable estimates among the others with the smallest bootstrap SEs. The bootstrap SEs of the LS estimates are larger than the others.

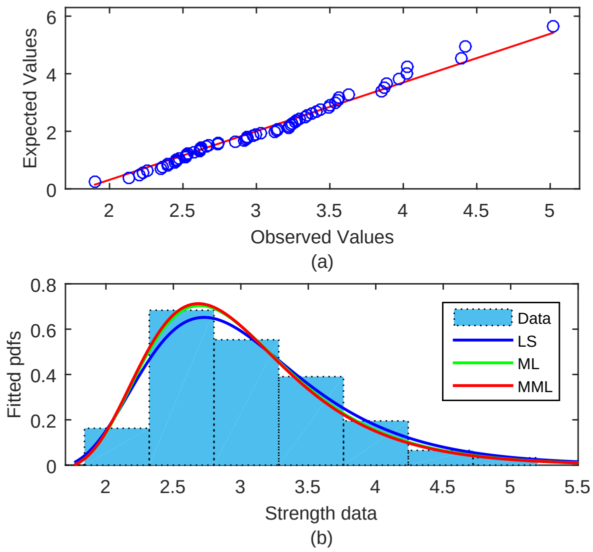

Figure 5: Q-Q plot of the strength data (a) and the fitted pdfs (b).

Table 6: The LS, the ML and the MML estimates of μ and σ.

	$\hat{\mu}$	$\hat{\sigma}$
LS	1.8701	0.5839
	(0.1351)*	(0.1002)*
ML	1.9009	0.5408
	(0.0895)*	(0.0537)*
MML	1.9009	0.5342
	(0.0878)*	(0.0485)*

*The values in parentheses are the bootstrap SEs.

7 Conclusion

In this study, we obtain the estimators of the location and the scale parameters of the SG distribution. For this purpose, we use the LS, the ML and the MML methodologies. Among these estimators, only the MML estimator provides the explicit estimators of the parameters of interest. The performances of the estimators are compared via Monte Carlo simulation study.

According to the simulation results, the usage of the LS estimator is preferable when $\eta = 2$. When $\eta = 3.5$, the joint efficiencies of the ML estimator and the MML estimator are better than the LS estimator for $n \leq 100$, however $n \geq 500$ vice versa. When

$\eta = 5$ and $\eta = 10$, the ML and the MML estimators outperform the LS estimator. The robustness properties of the estimators are also investigated. The LS estimators are found to be non-robust to the data anomalies (such as outliers, misspecification of the model etc.) as expected.

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