Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v10n3p629

Factors influencing the penalty cards in soccer

By Gómez–Déniz, Dávila–Cárdenes

Published: 15 November 2017

Electronic Journal of Applied Statistical Analysis
Vol. 10, Issue 03, November 2017, 629-655
DOI: 10.1285/i20705948v10n3p629

Factors influencing the penalty cards in soccer

Emilio Gómez–Déniz and Nancy Dávila–Cárdenes*

Department of Quantitative Methods and TiDES Institute. University of Las Palmas de Gran Canaria, Spain

Published: 15 November 2017

A compound Poisson distribution is used to study factors that motivate the fact of showing yellow and red cards in a football competition such as a national league, the FIFA World Cup or the UEFA Champions League. The resulting model is applied to the outcomes in the Spanish Football League during the season 2013–14, studying the partial and total effects on the home and away teams. It is shown that various factors, such as the victory of the guest team, goal difference in the game, total of fouls committed, attacking play of the home team, whether or not the match is a derby, round when the game is played, level of fair play, age of the referee, whether or not the referee has international experience, may influence the number of cards. The model introduced in this paper is simple and it works well, providing a useful tool that can be applied in different sport settings.

keywords: Compound Distribution, Football, Yellow Card, Red Card.

1 Introduction and motivation

In this paper, a parametric model to analyse the total number of cards (the sum of yellow and red cards, and only red cards) shown in a football competition such as a national league, the FIFA World Cup or the UEFA Champions League, is introduced. Football goes beyond the sport itself. The turnover generated around soccer has brought the interest in analysing different aspects that could affect the particular circumstances of the game. As a simple example, bettors can gamble on almost every aspect of the game. Among all the factors that might affect a match, we analysed the number of cards shown

*Corresponding author: nancy.davila@ulpgc.es

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

to players. The goal of this work is not intended to predict the total number of cards in a game, or to analyse the consequences of showing the cards, but put on the table what kind of factors could lead the referee to show yellow or red cards, beyond the inherent features of the game itself. When a player receives a card, his team will play with one less player during that match, or at a later stage, that player will be discarded for a future game due to the cards accumulated in different games. For instance, recently in the Spanish Cup, one team was eliminated from the competition for improperly aligning a player who did not fulfilled a suspension match due to the accumulated cards in his previous team. Undoubtedly, this issue had an important economic impact on his current club.

However, as it is said, rather than focus on the effects of the red and/or yellow cards, we study the factors that might explain the number of cards shown (both total and only red cards) in a football competition.

The effects of the cards shown have been analysed from diverse perspectives; see Ridder et al. (1994), Vecer et al. (2009), Anders and Rotthoff (2011), Reilly and Witt (2013) and Lex et al. (2015), among others. In addition, the question of the yellow and red cards shown, from the referee–behaviour standpoint, has been studied recently, see for example Boyko et al. (2007), Dawson et al. (2007), Dawson (2012), Dohmen (2008), Nevill et al. (2002). Apart from the economic effects that the cards shown have on the player, they also have a negative impact on the probability of winning games (see Anders and Rotthoff (2011)). This is so not only in the game where the card is issued to the player, but also in subsequent games due to the accumulated cards, the player perhaps will not play the following game. The results are useful to illuminate the behaviour of the actors involved, namely the players and the referees.

To the best of our knowledge, this is the first statistical model proposed in this setting. A discrete compound distribution is applied to analyse the factors that affect the total (yellow and red) and the red cards shown in the Spanish Football League. Let X be the random variable denoting the total number of cards shown in a competition and let Z_i be an indicator variable associated with the ith card shown, such that $Z_i = 1$ if the ith card is red, and $Z_i = 0$ if the ith card is yellow. Let now $Z = \sum_{i=1}^{X} Z_i$ be the total number of red cards shown, being therefore a compound distribution. When X follows a Poisson distribution and Z_i follows a Bernoulli distribution, the resulting distribution of Z is binomial. Clearly, $Z \leq X$ and the distribution of (X, Z) represents the joint distribution of total number of cards and of the red cards shown. This context seems to be appropriate for the statistical modelling of the total (red and yellow) and red cards shown in a competition, and therefore we study it in detail, obtaining the marginal and conditional distributions, and highlighting certain features of the distributions involved.

We present the results obtained from three regression models created for the dependent variables (X, Z): first the joint effect of home and away teams, then the same effects separately, first with relation to the home team, and then to the guest team. Finally, we examine the effect of the covariates on the same dependent variables. In each case, different factors were found to affect the outcome.

To place this problem in context, we first describe the laws governing disciplinary measures in football. The FIFA Disciplinary Code (2011) describes infringements of the rules, determines the sanctions incurred, regulates the organisation and function of the bodies responsible for taking decisions and stipulates the procedures to be followed before these bodies. The General Part of the First Title of the code describes the disciplinary measures to be taken during a match. Among the sanctions applicable (to persons, not institutions) are those of caution (yellow card) and expulsion (red card).

A yellow card is a warning issued by the referee to a player during a match to sanction unsporting behaviour of a less serious nature (see Law 12 of the Laws of the Game (2014), issued by the International Football Association Board). On receiving a red card, the player concerned, whether present from the start or playing as a substitute, is sent off and must leave the field immediately. This player cannot be replaced during the game, and so the team must continue with one less player. There are different causes of expulsion, but the most common is the receipt of two cautions during the same match (indirect red card) or the act of serious unsporting behaviour, as defined by Law 12 of the Laws of the Game, in which case the red card is regarded as direct.

If a player receives two cautions during the same match, he/she is automatically suspended from the next one in the same competition. Similarly, when a caution is received in two separate matches of the same FIFA competition, the player is automatically suspended from the next game in the same competition. Nevertheless, the Disciplinary Committee may exceptionally suspend or amend the application of this rule before the competition starts up; this happened, for instance, before the World Championships in 2010 and 2014, when FIFA decided that the accumulated yellow cards would not be taken into account from the start of the semi final stage.

An expulsion, a direct red card, automatically incurs suspension from the subsequent match, although the Disciplinary Committee may extend the duration of the suspension. The players are well aware of these sanctions, and in the case of key players, a suspension could have very important consequences for the team. For this reason, the players might play strategically in such a way during a match to force the imposition of a sanction, this would result in their suspension for a less important match, leaving them, subsequently, available to play a more relevant match.

The rest of this paper is organised as follows. The main model is developed in Section 2. In Section 3, together with a simple application to the FIFA World Cup held in Brazil in 2014, we present the model with the incorporation of covariates. The data and results on the Spanish Football League are described in Section 4 and conclusions are shown in the last Section.

2 Modelling the total and red cards

Although yellow and red cards are shown more frequently than in the past[1], during the ninety minutes of normal playing time, this event is uncommon in comparison with

[1]The first rules were established in 1863 by the Football Association of Great Britain, which published a document stipulating 14 official rules. In 1938, the International F.A. Board presented the revised Laws of the Game, in which the number of rules was expanded to 17. Finally, in 1997, a new modern version of the Laws of the Game of Football was defined, and the latest update of these Laws came into force on 1 June 2014.

others that occur during a match, such as fouls, corners, offside, etc. Accordingly (and except in the case of hot–tempered matches, in which large numbers of cards may be shown) the fact of showing a yellow, or red card, can be considered as a random variable that can be modelled by the Poisson distribution, which is habitually used to describe the number of events occurring within a given time interval. Thus, this distribution has been used for modeling the number of goals (see Greenhough et al. (2002)).

Therefore, we assume that the number of yellow and red cards shown (X) in a given time interval has a Poisson distribution, with the parameter $\lambda > 0$ and a probability function that is given by

$$\Pr(X = x) = \exp(-\lambda)\frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

When a referee shows a card to a player, it may be yellow or red. Let us now introduce a second random variable, separating the card–showing event into two sub–events (show yellow card and show red card), as follows. Let Z_i be an indicator variable associated with the ith card shown, such that $Z_i = 1$ if the ith card is red, and $Z_i = 0$ if the ith card is yellow. Furthermore, we assume that the probability function of Z_i is given by

$$\Pr(Z_i = z_i) = \begin{cases} p, & \text{if} \quad z_i = 1, \\ 1-p, & \text{if} \quad z_i = 0, \end{cases}$$

where $0 < p < 1$. In other words, we assume that Z_i follows a Bernoulli distribution.

Now, let $Z = \sum_{i=1}^{x} Z_i$ be the number of red cards shown. If Z_i ($i = 1, \ldots, x$) are assumed to be mutually independent, then the conditional probability function of Z, given $X = x$, is binomial with parameters x and p. That is,

$$\Pr(Z|X = x) = \binom{x}{z} p^z (1-p)^{x-z}, \quad z = 0, 1, \ldots, x.$$

Thus, the conditional mean and the variance are given by $E(Z|X = x) = xp$ and $var(Z|X = x) = xp(1-p)$. This means that the regression of Z on X and the conditional variance of Z, given X, are linear.

Therefore, the joint distribution of the total number of cards shown (X) and of the red cards shown (Z) has the probability function

$$\begin{aligned} \Pr(X = x, Z = z) &= \Pr(Z|X = x)\Pr(X = x) \\ &= \frac{\lambda^x p^z (1-p)^{x-z}}{z!(x-z)!}\exp(-\lambda), \end{aligned} \qquad (1)$$

for $x = 0, 1, \ldots,$ $z = 0, 1, \ldots, x$. From the joint probability function given in (1) we obtain $\Pr(Z = z)$, i.e. the marginal probability function of the number of red cards, given by

$$\Pr(Z = z) = \frac{(\lambda p)^z}{z!}\exp(-\lambda p), \quad z = 0, 1, \ldots, \qquad (2)$$

for which the mean and variance are the same and given by

$$E(Z) = var(Z) = p\lambda. \tag{3}$$

The probability function in (2) is a Poisson distribution with parameter λp. Under the model proposed, if the total number of cards shown is Poisson with mean λ, then the total red cards shown is again Poisson, with mean λp, which is the product of the mean of the total number of cards and the probability of a red card being shown. This seems reasonable; thus, the mean of the red cards shown is a proportion of the total number of cards shown.

Another way to obtain the expression given in (2) is the following. Consider a competition, for which X is the random variable giving the number of cards (yellow or red) shown in the competition, and Z the random variable giving the number of players cautioned with red cards by the end of the competition. If each player has a probability of p of being shown a red card, then, $\Pr(Z = z)$ and $\Pr(X = x)$ are related as follows:

$$\Pr(Z = z) = \sum_{x=z}^{\infty} \binom{x}{z} p^z (1-p)^{x-z} \Pr(X = x). \tag{4}$$

It is evident that (4) represents a map from probability function to probability function. That is, $\sum_{z=0}^{\infty} \Pr(X = x) = \sum_{Z=0}^{\infty} \Pr(Z) = 1$ and $\Pr(z) \geq 0$, $z = 0, 1, \ldots$

Then, if X follows a Poisson distribution with parameter $\lambda > 0$ we have

$$\begin{aligned}
\Pr(Z = z) &= \sum_{x=z}^{\infty} \binom{x}{z} p^z (1-p)^{x-z} \frac{\lambda^x}{x!} \exp(-\lambda) \\
&= \frac{\exp(-\lambda)}{z!} \left(\frac{p}{1-p}\right)^z \sum_{x=z}^{\infty} \frac{[\lambda(1-p)]^x}{(x-z)!} \\
&= \frac{\exp(-\lambda)}{z!} \left(\frac{p}{1-p}\right)^z \sum_{j=0}^{\infty} \frac{[\lambda(1-p)]^{j+z}}{j!} \\
&= \frac{(\lambda p)^z}{z!} \exp(-\lambda p), \quad z = 0, 1, \ldots
\end{aligned}$$

Expression (4) can be viewed as a weighted sum of binomial probabilities where the weights are given by the probability of a card being shown, $\Pr(X = x)$. More specifically, it is the mean of total number of cards shown conditional to the fact of $X = x$ cards (yellow and red) shown being given by xp and assuming the existence of a heterogeneity factor which causes a player to be shown a card. Hence, expression (4) can be viewed as a mixture distribution. From this standpoint, the model provides a framework in which random effects are incorporated into the Poisson assumption.

To determine the correlation between X and Z, we need $E(XZ)$, which is found to be $E(XZ) = p\lambda(1 + \lambda)$. Then the covariance is $cov(X, Z) = p\lambda$, which is always positive.

This is coherent with the fact that total number of cards shown and red cards shown are positively correlated. The coefficient of correlation is $\rho = \sqrt{p}$.

A similar model, in the setting of traffic accidents, was proposed by Leiter and Hamdan (1973). See Cacaoullos and Papageorgiou (1982) for details.

Clearly, the number of red and yellow cards shown may be influenced by different characteristics and variables, and explanatory variables may be useful to explain this setting and this is examined in the next section.

3 Normalising the distribution

For the sake of convenience, we rewrite (1) in another form, so that covariates may be introduced later into the model. By equating (3) to μ_2 and the mean of the marginal probability function of X to μ_1, that is $\lambda = \mu_1$, we obtain the normalised joint distribution of (X, Z) from (1), as follows:

$$\Pr(X = x, Z = z) = \frac{\mu_2^z(\mu_1 - \mu_2)^{x-z}}{(x - z)!z!} \exp(-\mu_1), \tag{5}$$

for $x = 0, 1, \ldots, z = 0, 1, \ldots, x$, $\mu_1 > 1$ and $\mu_2 > 0$.

This probability function (5) satisfies the condition that the marginal means should be given by $E(X) = \mu_1$ and $E(Z) = \mu_2$ and thus is appropriated for introducing covariates. In order to show how the distribution (5) takes effect, we provide an example addressing the total number of cards and the red cards shown in the FIFA World Cup competition held in Brazil in 2014.

Example 1 (An example from the FIFA World Cup)

Consider the total yellow and red cards shown in the FIFA World Cup 2014 competition held in Brazil. A total of 32 teams took part in the final stage of the tournament. In the first round, or group stage, the 32 teams were divided into eight groups of four, each of which engaged in a round–robin tournament. Thus, each team played 3 matches, and the two highest ranked teams in each group advanced to the knockout stage. The knockout stage started with 16 teams, and consisted of four rounds, from the last 16, through the quarter finals and semi finals, to the final (plus the third place play–off). A total of 64 matches were played during the competition, which took place from 12 June to 13 July.

With respect to yellow and red cards, the following rules were applied during the tournament. If a player received a red card (direct or indirect), he was automatically suspended for the next match. Any player sanctioned with two yellow cards in two different games was suspended for the next game. This rule was applied from the beginning of the group stage until the end of the quarter final matches, at which point all single yellow cards were cancelled. This rule change was made in order to give the players a better chance to compete in the final, if their team got that far in the competition. In order to apply our model, data on the red and yellow cards shown were obtained from `http://www.uefa.com/`.

The empirical values of the data are shown in bold in Table 1. The total number of yellow and red cards shown in the competition in the data were 198 and 10, respectively. This last corresponds to the direct red card shown to a player and the accumulation of two yellow cards shown to the player. Figure 1 is a plot of the evolution of the total number of cards and red cards shown in this competition, revealing a large concentration of yellow cards shown in the latest matches. In other words, the number of cards shown increased as the tournament progressed. In these later stages, obviously, fewer matches took place (due to the knockout nature of the competition). One factor that might explain this paradoxical situation is the fact that cards were cancelled after the quarter finals, and so players whose teams remained in the competition after the initial stage might then be more prepared to run the risk of receiving a card.

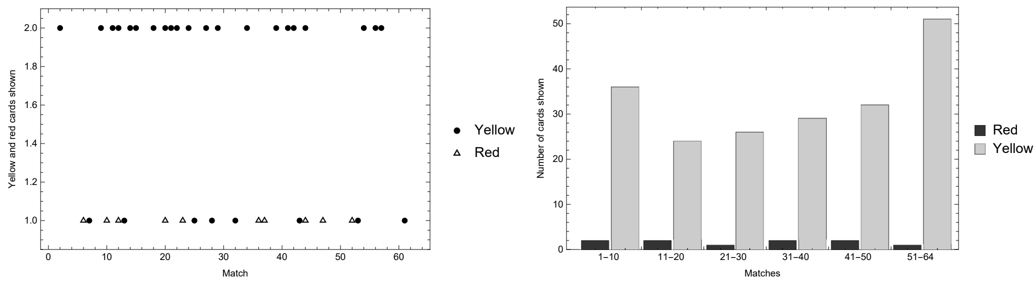

Figure 1: Plot of the evolution of yellow and red cards shown in the FIFA World Cup 2014 (left) and number of yellow and red cards shown (right)

In this study, the maximum likelihood method (see Appendix) was used to estimate the two parameters of the bivariate distribution in (5), from the appropriated likelihood equations. The estimated values obtained are given by $\widehat{\mu}_1 = 3.09375\,(0.219863)$ and $\widehat{\mu}_2 = 0.15625\,(0.0494106)$, where the corresponding standard errors are shown in brackets. These estimated parameters were used to compute the expected values by using expression given in (5). The value obtained for the Akaike information criterion (AIC) was 296.355. Note that AIC $= 2(k - \ell_{\max})$, where k is the number of model parameters and $\ell_{\max}$ is the maximum value of the log–likelihood function, see Akaike (1974) for details. The goodness–of–fit was determined by standard Pearson's chi squared test statistics, given by

$$\chi^2 = \sum_{i=1}^{n_0} \frac{(\text{observed}_i - \text{expected}_i)^2}{\text{expected}_i}$$

with the following grouping procedure: the outermost classes were consolidated to produce theoretical class sizes of 5 or larger. It is known that χ^2 follows a chi–squared distribution with $n_0 - k - 1$ degrees of freedom, where n_0 is the number of classes considered in order to compute the value of χ^2. In the present case, we obtained $\chi^2_3 = 4.18$ (which is well below 7.81, the 5% critical value of χ^2 with 3 degrees of freedom) with a p-value of 24.26%. From the estimated values of the parameters, the estimated correlation obtained from the expression $\rho = \sqrt{\widehat{\mu}_2/\widehat{\mu}_1}$, is 0.224733, versus the true value of 0.37478. The fitted values are also shown in Table 1, below the sample values in bold.

A graph showing the number of red cards expected to be shown, given the total number of cards shown, was plotted (see Figure 2). Clearly, the observed values follow a pattern that is almost linear, and which is well described by the expression $E(Z|X=x) = x\frac{\widehat{\mu}_2}{\widehat{\mu}_1}$. In view of this satisfactory result, we believe the distribution given in (5) is suitable for the data set in question.

Table 1: 2014 FIFA World Cup disciplinary record in terms of yellow and red cards shown in total matches played. Empirical values above and fitted values below

Red Cards \ Total Cards	0	1	2	3	4	5	6	7	8	9	10	Total
0	**0**	**8**	**18**	**13**	**9**	**4**	**1**	**1**	**0**	**0**	**0**	**54**
	2.90	8.52	12.52	12.25	9.00	5.29	2.60	1.08	0.40	0.13	0.04	54.73
1	**0**	**0**	**3**	**1**	**2**	**1**	**1**	**0**	**1**	**0**	**1**	**10**
	0.00	0.45	1.33	1.95	1.91	1.41	0.82	0.40	0.17	0.06	0.02	8.52
Total	**0**	**8**	**21**	**14**	**11**	**5**	**2**	**1**	**1**	**0**	**1**	**64**
	2.90	8.97	13.85	14.20	10.91	6.70	3.42	1.48	0.57	0.19	0.06	63.25

3.1 Including covariates

In this section, we investigate the effect of covariates that account for the total number of cards and the red cards shown. Obviously, some factors must be considered of especial importance in explaining the endogenous variates (X_i, Z_i). Expression (5) is useful for this purpose, since the marginal means are μ_{1i} and μ_{2i}. Two appropriate links are needed to connect the covariates with these marginal means. A natural way to do this is to assume that (X_i, Z_i) for $i = 1, \dots, n$ follows the probability function given in (5) and

$$\begin{aligned} \log \mu_{1i} &= \omega_{1i}\beta_1, \\ \mu_{2i} &= \frac{\mu_{1i}\exp(\eta_{2i}\beta_2)}{1+\exp(\eta_{2i}\beta_2)}, \end{aligned}$$

where ω_{1i} and η_{2i} denote vectors of m explanatory variables for the ith observation, i.e. with components ω_{ji} and η_{ji}, , $(j = 1, \dots, m)$, used to model μ_{1i} and μ_{2i}, respectively, and where $\beta_k = (\beta_{k1}, \dots, \beta_{km})'$, $(k = 1, 2)$ denotes the corresponding vector of regression coefficients. The log–linear specification for μ_{1i} is widely used, while the link function for μ_{2i} was chosen in this way to ensure that the latter would not be larger than μ_{1i} and thus would be compatible with $Z \le X$.

The mean number of red and yellow cards shown may be influenced by several characteristics and variables, and the explanatory variables that are used to model each

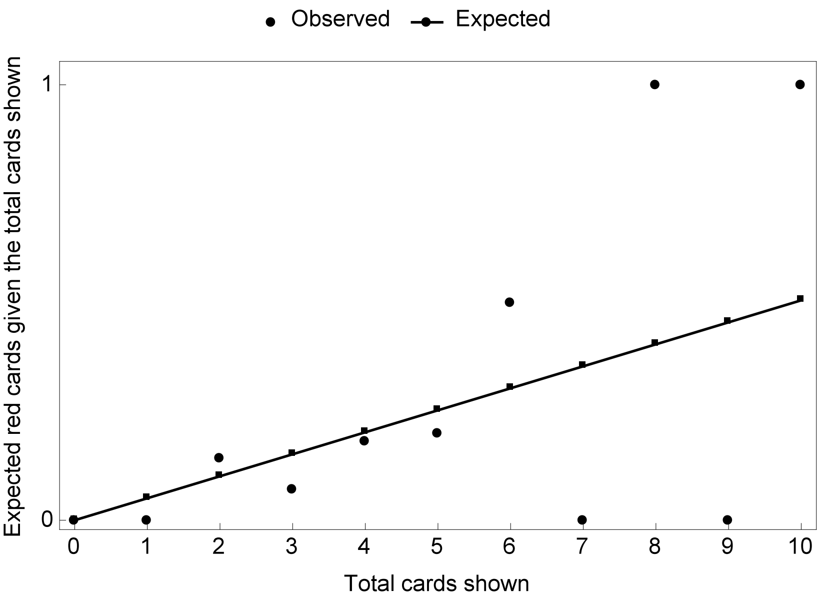

Figure 2: Expected red cards shown given the values of the total number of cards shown

parameter μ_{1i} and μ_{2i} may not be the same in practice. In respect of this, Cameron and Trivedi (1998) provide good overviews of standard count regression models.

Maximising the log–likelihood function (6) with respect to β_k $(k = 1, \dots, m)$ is straightforward via Newton–Raphson iteration or the scoring algorithm. The solutions of the nonlinear equations shown in the Appendix provide the maximum likelihood estimates of these parameters. However, these equations cannot be solved explicitly, but must be addressed either by a numerical method or by directly maximising the log–likelihood function. Different initial values of the parametric space can be considered as a seed point. In this study, the FindMaximum function of Mathematica software package v.10.0 (Wolfram (2003)) was used, although the same results can be obtained by other methods, too, such as Newton, PrincipalAxis or QuasiNewton (all of which are available in this package), or by other packages such as R, Matlab or WinRats. Finally, the standard errors of the parameter estimates were obtained by inverting the Hessian matrix. The Fisher information matrix is shown in closed–form expression in the Appendix.

3.1.1 Marginal effects and interpretation of the coefficients

The marginal effect reflects the variation of the conditional mean of X and Z due to a one–unit change in the j-th covariate, and is calculated as

$$\begin{aligned}
\frac{\partial \mu_{1i}}{\partial \beta_{1j}} &= \omega_{ji} \exp(\omega_{1i}\beta_1) = \omega_{ji}\mu_{1i}, \\
\frac{\partial \mu_{2i}}{\partial \beta_{2j}} &= \eta_{ji}\mu_{2i}\left(1 - \frac{\mu_{2i}}{\mu_{1i}}\right),
\end{aligned}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. Thus, the marginal effect indicates that a one–unit change in the j-th regressor increases or decreases the expectation of total number of cards and red cards being shown depending on the sign, positive or negative, of the regressor for each mean. For indicator variables such as ω_{ik}, which takes only the value 0 or 1, the marginal effect in term of the odds–ratio is $\exp(\beta_{1j})$ for μ_{i1} and $\exp(\beta_{2j})$ for μ_{i2}. Therefore, the conditional mean is $\exp(\beta_{1j})$ times larger if the indicator variable is one rather than zero in the case of μ_{i1}, and a similar conclusion is drawn for μ_{i2}.

4 Application to the Spanish football league

Twenty teams play in the first division of the Spanish Football League (known as the LFP, from its Spanish initials, or La Liga). The competition follows a two–round format in which each team plays each other twice, once at home and once away. Three points are awarded for a win, one for a draw and none for a loss. The total points won by the end of the competition determine the final ranking. The highest ranked club becomes the champion. If at the end of the tournament two or more clubs have the same number of points, the champion team may be determined by application of the Fair Play scales, although in fact this possibility is the last in the list of tie–breakers described in the rules. Statistics and data for all the matches played are available on the LFP website.

4.1 Data

According to the UEFA coefficient, the Spanish Football League has been the strongest competition in Europe over the past five years. The basis for the UEFA coefficient is the performance of teams in the European club competitions over a five–year period, for which each team is awarded two points for a win and one for a draw, plus a coefficient of 20% of the clubs' association ranking.

The Spanish clubs have won the most UEFA Champions League tournaments and one Spanish club is the most successful in the UEFA Europa League. Rivalry is fierce in La Liga, especially among the two top–ranked clubs in Europe, Real Madrid and F.C. Barcelona. Every match between these top teams is considered one of the main sports events not only in Spain, but also around the world. It is estimated that over 400 million people watched their last match, and according to the company that owns the television rights, "There is no football match between teams from the same country that arouses greater international interest". In Spain, not only the matches between these two teams are considered of special significance, but also those played against any of the other three top–ranked clubs in Spanish football, Atlético de Madrid, Sevilla and Valencia. Other major games are those played between teams from the same city or region, such as Atletico de Madrid vs Real Madrid or Athletic de Bilbao vs Real Sociedad. Such derbies are played with special intensity and higher numbers of yellow and red cards are often shown.

The data used in this analysis were obtained from the following website

`http://www.football-data.co.uk`

which compiles and presents data from every match, including the winning team, total goals scored, total fouls, the stage of the league competition (first or second round), the total number of shots, the number of shots on target, the number of yellow and red cards shown, the number of corners and of offside and how many times the team's shots hit the post. All data are compiled for the home and away team in every match. Among these data, the factors we believe most relevant to the fact of showing yellow and red cards are included and analysed in the following section. The total number of yellow and red cards shown in the competition in the data were 1981 and 115, respectively.

In the first case, without covariates, the maximum likelihood method was used to estimate the parameters of the bivariate distribution in (5), from the corresponding likelihood equations shown in the Appendix. The estimated values obtained are given by

$$\widehat{\mu}_1 = 5.2132\,(0.1171), \quad \widehat{\mu}_2 = 0.3026\,(0.0282),$$

where the standard errors are shown in brackets. The maximum value of the log–likelihood function was –1085.89. The chi squared test statistic obtained was $\chi^2_{13} = 7.15$ (below 22.36, the 5% critical value of χ^2 with 13 degrees of freedom) with a p-value of 89.41%. From the estimated values of the parameters, the estimated correlation obtained from the expression $\rho = \sqrt{\mu_2/\mu_1}$, is 0.2410, in comparison to the true value of 0.35753. The fitted values are also shown in Table 5, below the sample values in bold. Needless to say that a different grouping will give us different values of the χ^2 statistics. Nevertheless, looking at Table 5, it seems that there is no deviations from the model in any part of the table, and examination of the ungrouped table confirms this and the reader can also see the marginal results. In view of the good results obtained, we believe the distribution given in (5) is appropriated for this data set.

4.2 Variables

In our analysis, to determine what factors might affect the number of yellow and red cards shown, the variables considered were grouped into four categories: those related to the game statistics (RFT2, FGOALS, TF, HAPVF, AS AF and AY), those we term match variables (DERBY, ROUND and FP), extra–game variables (BUDGET and SH) and those associated with the referee (AGER and INTERNATIONAL).

The following variables were included in the game statistics: RFT2, the final result of the match, awarded a value of 1 if the away team wins and 0 otherwise; FGOALS, measures the square root of the absolute value of the difference between the goals scored by the home team (FTHG) and the away team (FTAG) after the full time match[2]. This variable was also computed as a categorical variable, and the results obtained are similar to those of the variable FGOALS considered here; TF, the total fouls committed; HAPVF, the ratio between the home team's total shots plus on–target shots and the

[2]This is a standard transformation in soccer analysis, see for instance Gelman: "Stan Goes to the World Cup" on his blog) since larger score differences are expected to make progressively less difference on the impact of the cards awarded.

number of fouls committed by the away team, i.e., (HS+HST)/AF, this variable provides a measure of the degree of attacking play offered by the home team; the same ratio was calculated for the away team, but it was not statistically significant; AS, the total shots made by the away team; and finally, AF, representing the fouls committed by the away team, and AY, representing the yellow cards shown to the away team. Obviously, the number of illegal actions committed will have a direct influence on the number of yellow and red cards shown.

There are two match variables: DERBY, which takes the value 1 when the match is played between teams from the same city or region or between the strongest teams in La Liga, and 0 otherwise; and ROUND, which takes the value 1 when the match is played in the second round of the league competition, and 0 when it is played in the first. In addition to these, we take into account the variable fair play, FP, which is the ratio between the fair play points awarded during the previous year for the home and away teams (these points are awarded in accordance with the yellow and red cards shown).

The extra–game variables used were BUDGET, defined as the absolute value of the difference between the budgets of the home team, BH, and the away team, BA; and SH, referring to the capacity of the home stadium.

Finally, the variables related to the referee were his age, AGER, and whether he had international experience, INTERNATIONAL, which was scored as 1 if the referee had such experience, and 0 otherwise. Other variables related to the referee were PROFESSION (a measure of idiosyncrasy between the referees) AND YEARS OF EXPERIENCE (number of years of experience in the first division). However, these last two ones never resulted significant, in consequence, they were not taken into account.

A brief description of the variables considered is shown in Table 2.

Table 3 presents descriptive statistics of the continuous variables considered, and Figure 5 shows the corresponding Box–Whisker charts. These charts show that except for the TF variable, there are outliers for all the other variables, which suggests that perhaps these variables should be converted to dichotomous variables, by splitting the sample at some point on the scale(s) of measurement. This is a matter that lies beyond our present scope and may be suitable for future study.

As in the previous example, Figure 3 provides a plot of the evolution of total number of cards and of red cards shown in the Spanish football competition considering home and away teams together, and separately. Figure 4 reveals a large concentration of cards (yellow and red) shown in the last matches.

4.3 Results

Table 4 shows the joint effect for the home and away teams in terms of the total number of cards. The following significant variables for the season 2013–2014 were obtained: the game statistics variables, RFT2, FGOALS and TF; the match variable, DERBY. The marginal effect is positive for all the variables except for FGOALS, what means that the expectation of showing cards decreases with the absolute value of the difference between the goals scored by the home and away teams, i.e., when one team wins by a considerable margin, there is a lower probability of showing cards. In other words, in

Table 2: Description of the variables

Variable	Description
<u>**Game statistics**</u>	
RFT2	Scored as 1 if the away team wins and 0 otherwise.
FGOALS	Square root of the absolute value of the goals difference between home (FTHG) and away (FTAG) teams after the full time match.
TF	Total fouls.
AS	Away team shots.
AF	Number of away team fouls.
AY	Away team yellow cards.
HAPVF	The sum of home shots and home shots on target divided by away fouls $(\mathrm{HS}+\mathrm{HST})/AF$.
<u>**Match variables**</u>	
DERBY	Match played between teams from the same city or region or between the strongest teams in the league.
ROUND	Scored as 0 if the match is played in the first round and 1 in the second.
FP	The ratio between the previous year's fair play points for the home team and those for the away team.
<u>**Extra games**</u>	
BUDGET	Absolute value of the difference between the home team budget and that of the away team divided by 100.
SH	Capacity of the home stadium.
<u>**Referee**</u>	
AGER	Age of the referee.
INTERNATIONAL	Scored as 0 if the referee has no international experience and 1 if he does.

Table 3: Descriptive data for the continuous variables considered

	BUDGET	FGOALS	TF	HAPVF	FGOALS	AS	AY	AGER
Maximum	568	7	43	10	6	35	7	45
75%	85	2	32	1.78	1	14	3	43
Median	25	1	28	1.33	1	11	3	40
25%	12	1	24	1.00	0	8	2	37
Minimum	0	0	13	0.35	0	2	0	30

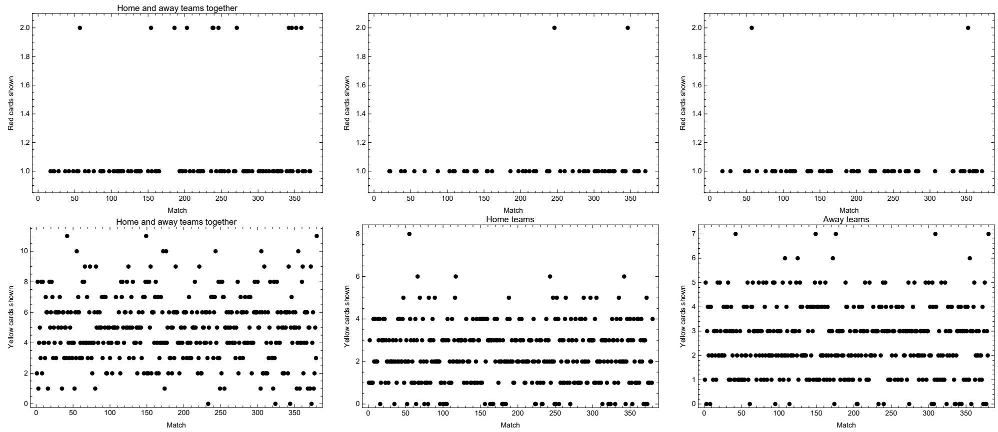

Figure 3: Evolution of yellow and red cards shown during the league competition. Home and away teams together (left), home team (center) and away team (right)

the absence of competitiveness (a large goal difference is interpreted as representing a difference in quality between the teams), as a result the mean of the number of cards shown decreases.

With respect to the variables TF (total fouls) and AGER (the age of the referee); the higher is the numbers of fouls committed and the older is the referee, the greater is the expectation of the number of cards shown. Obviously, cards may be shown when fouls are committed. However, it less clear the meaning of the positive marginal effect for the AGER variable, it might be considered as the older referees are less tolerant, that is, as they are more experienced, they are more inclined to impose their authority and more favourably disposed to issue cards.

The probability of showing cards is also greater when the match is a derby, because these games are often played with greater intensity. In addition, when the final score is favourable to the away team, this increases the mean of the total number of cards shown.

With relation to the number of red cards, the round of the competition in which the match was played was found to be significant, with a positive marginal effect. Thus, when the match is played in the second round of the league competition, a red card is more likely to be shown. We interpret this finding as follows: once the first round has been completed, an initial ranking of the teams is established and the results of the second round will then determine the final options of the teams, i.e. whether they have any chance of qualifying to play in Europe, or whether they are candidates for relegation. Therefore, in this second stage of the competition, the teams probably play more intensely and they will receive red cards. A similar situation arises in tournaments such as the World Cup, where we observed that the mean of the number of red cards awarded increased as the competition progressed.

With respect to the total fouls, TF, the more fouls are committed, the higher is the number of red cards shown. This outcome is logical, and in accordance with the rules of

Table 4: Home and away teams together. 2013–14 Spanish League disciplinary record in terms of yellow and red cards shown in total matches played

		Total cards			
Variable	Estimate	Standard error	$\lvert t\rvert$-statistic	Pr > $\lvert t\rvert$	Marginal effect
Game statistics					
RFT2	0.1219	0.0499	2.4447	0.015	0.636
FGOALS	–0.1417	0.0361	3.9206	< 0.001	–0.738
TF	0.0263	0.0038	6.7667	< 0.001	0.137
Match variables					
DERBY	0.1019	0.0565	1.8030	0.072	1.107
Referee					
AGER	0.0084	0.0053	1.5829	0.114	0.044
		Red cards			
Variable	Estimate	Standard error	$\lvert t\rvert$-statistic	Pr > $\lvert t\rvert$	Marginal effect
Game statistics					
TF	0.0326	0.0197	2.5491	0.011	0.009
HAPVF	0.2989	0.1390	2.1508	0.032	0.085
Match variables					
ROUND	0.5079	0.1992	2.5491	0.011	1.661
Extra game					
BUDGET	–0.1640	0.0749	2.1887	0.029	–0.047
Other variables					
Constant (total number of cards)	0.6559	0.2280	2.8760	0.004	
Constant (red cards)	–4.3068	0.7471	5.8066	< 0.001	
Value of the Akaike information criterion 2093.25					

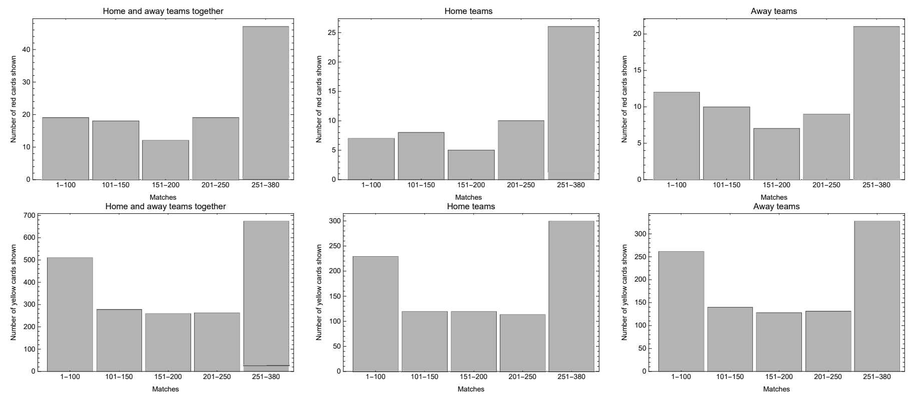

Figure 4: Number of yellow and red cards shown during the league competition. Home and away teams together (left), home team (center) and away team (right)

the game.

The results for the variable HAPVF means that, the higher the ratio between the sum of shots and shots on target by the home team and the total number of fouls committed by the away team, the greater is the probability of showing red cards. As commented above, this variable reflects the home team's ability to play in an attacking style. It means that when the home team plays near the box of its opponent, a higher number of red cards tend to be shown.

Finally, when a joint effect on the home and away teams is considered, a large difference in their budgets tends to reduce the number of red cards shown. The financial resources for the teams are assumed to be related to the quality of the players available to each team. When there is a large difference between the budgets of the teams, there is less competitiveness and hence fewer red cards are shown.

Again, a goodness–of–fit measure for the models with covariates is given in Table 5, and below the fitted values obtained for the model without covariates. The fitted cell frequencies were calculated in this case using the method proposes by Kim (2013). Let $\widehat{p}(c_{1i}, c_{2i})$, $i = 1, 2, \ldots, n$, $c_{1i}, c_{2i} = 0, 1, \ldots$ denoted the fitted probability that (X_i, Z_i) has (c_1, c_2), then the fitted frequency in cell (c_1, c_2) is computed as $\sum_{i=1}^{n} \widehat{p}(c_{1i}, c_{2i})$. The chi squared test statistic obtained was $\chi_4^2 = 12.67$ (below 13.28, the 1% critical value of χ^2 with 4 degrees of freedom).

In order to contrast if there exists any difference between the covariates affecting the total number of cards in the joint analysis and those that may affect to the home and away team considered separately. As we will see there are some covariates that affect to the away team and not to the home team, for instance, the variable DERBY, or viceversa, as the total fouls, TF.

When only the home team is considered (see Table 6) the following results were obtained. For the total number of cards shown, we observe the same significant variables as for the joint distribution, except DERBY, which does not have a significant factor for the home team. However, the higher the FP (the fair play ratio) is for the home team,

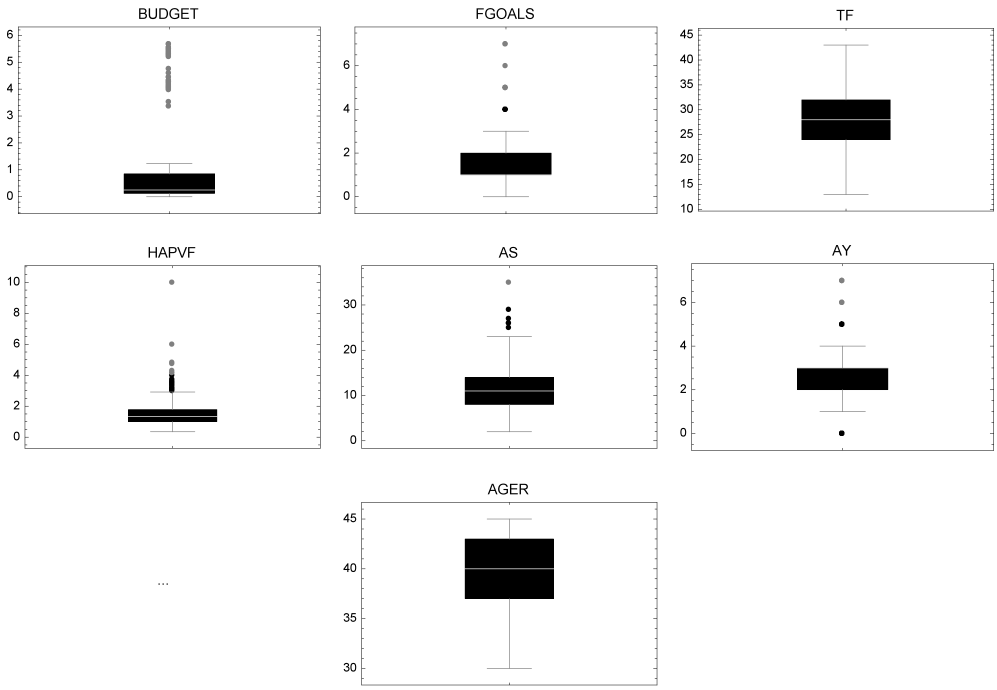

Figure 5: Box–Whisker chart for the continuous variables considered

the greater is the expectation of total number of cards. This finding is in line with the definition of FP, because a higher FP ratio tends to reflect an absence of fair play.

With regard to the number of red cards shown, ROUND and BUDGET were the only variables that remained significant for the home team. Thus, TF (total fouls) and HAVPF (the attacking play of the home team) vanished while RFT2 and INTERNATIONAL appeared. Thus, when the away team wins, this increases the number of red cards shown to the home team. With relation to the covariates connected to the referee, we highlight INTERNATIONAL, since it implies that when the referee has international experience the expectation of the number of red cards being shown to the home team increases.

In a separate analysis for the away team (see Table 7), the DERBY variable was found to be significant for the joint distribution, it means that in this kind of games, the away team is more likely to receive yellow and/or red cards than when the two teams are considered jointly. As noted previously, this variable was not significant in the separate analysis for the home team.

The variable FGOALS appears also in this separate analysis, as it occurred in the previous. Lastly, for the total number of cards, a new variable, AF, arises. Here, a positive marginal effect was recorded, meaning that an increase in the number of fouls committed by the guest team, it implies an increase in the probability of the total number of cards received.

Table 5: 2013–14 Spanish League disciplinary record in terms of yellow and red cards shown in total matches played. Empirical values above and fitted values below, without and with covariates, respectively

Red Cards \ Total Cards	0	1	2	3	4	5	6	7	8	9	10	11	12	Total
0	**4**	**14**	**21**	**36**	**56**	**46**	**54**	**13**	**21**	**7**	**3**	**2**	**0**	**277**
	2.07	10.16	24.94	40.83	50.12	49.22	40.29	28.26	17.35	9.46	4.65	2.07	0.85	280.27
	3.51	13.97	29.10	42.16	47.75	44.99	36.66	26.49	17.29	10.33	5.71	2.94	1.42	282.23
1	**0**	**0**	**1**	**11**	**4**	**18**	**17**	**17**	**11**	**5**	**3**	**3**	**1**	**91**
	0	0.63	3.07	7.55	12.35	15.17	14.90	12.19	8.55	5.25	2.86	1.41	0.63	84.56
	0	0.83	3.41	7.37	11.11	13.10	12.84	10.87	8.15	5.51	3.41	1.94	1.03	79.57
2	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**3**	**1**	**4**	**0**	**2**	**0**	**12**
	0.00	0.00	0.09	0.46	1.14	1.87	2.30	2.25	1.84	1.29	0.79	0.43	0.21	12.67
	0.00	0.00	0.12	0.51	1.14	1.76	2.16	2.19	1.92	1.49	1.05	0.67	0.40	13.48
Total	**4**	**14**	**22**	**47**	**60**	**65**	**72**	**33**	**33**	**16**	**6**	**7**	**1**	**380**
	2.07	10.79	28.10	48.84	63.61	66.26	57.49	42.70	27.74	16.00	8.30	3.91	1.69	377.50
	3.74	15.20	32.79	49.76	59.48	59.44	51.49	39.62	27.53	17.49	10.27	5.62	2.88	375.31

A separate analysis of the number of red cards for the away team gives the following results. The larger is the number of shots made by the away team, the lower is the probability of red cards received; while the older is the age of the referee, the higher is the probability of red cards received.

5 Conclusions and further research

In this paper, we have presented a flexible bivariate count data regression model based on a compound distribution to represent the cards shown (both total yellow and red, and only red cards) in football matches. The model proposed could be extended to accommodate truncated, censored and zero–inflated correlated count data models in the same framework. We have obtained a computationally straightforward and simple closed form bivariate model. Our empirical application shows that the proposed model provides a good fit to the data for the FIFA World Cup and the Spanish League for the season 2013–14.

It is well known that a good portion of red cards in a football competition is the result of the second yellow card shown to a player, and the model proposed here does not reflect this fact, although assume dependence between the total cards and the yellow cards shown. A more rational model should consider to replace the distribution (1) by the quasi binomial distribution (see Consul (1990)). Under this model Z would represent the number of red cards shown when X cards (yellow and red) have been shown. In such a case, the probability of the first red card is p and the probability of showing a red card in the following is $p + x\phi$, being therefore proportional to the number of total cards shown. Since ϕ can take negative and positive values, a positive value of ϕ would reflect large probability to the event red card when the yellow cards shown is also large.

Table 6: Home team. 2013–14 Spanish League disciplinary record in terms of yellow and red cards shown in total matches played

	Total cards				
Variable	Estimate	Standard error	$\lvert t\rvert$-statistic	Pr $> \lvert t\rvert$	Marginal effect
Game statistics					
RFT2	0.2157	0.0732	2.9462	0.003	0.530
FGOALS	–0.1566	0.0539	2.9015	0.004	–0.385
TF	0.0292	0.0057	5.1226	< 0.001	0.072
Match variables					
FP	0.1686	0.0805	2.0941	0.037	0.415
Referee					
AGER	0.0170	0.0078	2.1789	0.030	0.042
	Red cards				
Variable	Estimate	Standard error	$\lvert t\rvert$-statistic	Pr $> \lvert t\rvert$	Marginal effect
Game statistics					
RFT2	1.2994	0.2941	4.4176	< 0.001	0.179
Match variables					
ROUND	0.8927	0.3017	2.9589	0.003	2.441
Extra game					
BUDGET	–0.2781	0.1226	2.2679	0.024	–0.038
Referee					
INTERNATIONAL	–0.5918	0.2962	1.9983	0.046	0.553
Other variables					
Constant (total number of cards)	–0.7018	0.3418	2.053	0.041	
Constant (red cards)	–3.2880	0.3274	10.0410	< 0.001	
Value of the Akaike information criterion 1560.85					

Table 7: Away team. 2013–14 Spanish League disciplinary record in terms of yellow and red cards shown in total matches played

Variable	Estimate	Standard error	$\|t\|$-statistic	Pr > $\|t\|$	Marginal effect
Total cards					
Game statistics					
FGOALS	–0.1177	0.0475	2.4794	0.013	–0.324
AF	0.0426	0.0073	5.8250	< 0.001	0.117
Match variables					
DERBY	0.1445	0.0769	1.8790	0.061	1.155
Red cards					
Variable	Estimate	Standard error	$\|t\|$-statistic	Pr > $\|t\|$	Marginal effect
Game statistics					
AS	–0.0949	0.0336	2.8237	0.005	–0.013
Referee					
AGER	0.0614	0.0355	1.7291	0.085	0.009
Other variables					
Constant (total number of cards)	0.4986	0.1225	4.0704	< 0.001	
Constant (red cards)	–4.2722	1.4667	2.9127	0.004	
Value of the Akaike information criterion 1655.30					

On the other hand, the maximal number of red cards that can be shown to players on the field is limited to 4 per team, otherwise the game is terminated. Thus, a future line of research could be to derive a model where the Z variable would include an upper bound support.

Finally, it is apparent that the number of red cards shown in a competition has a large number of zero values. Thus, a similar model to the one proposed here where the Z variable were inflated with zeros would be more practical.

Acknowledgement

The authors would like to express their gratitude to two anonymous referees for their relevant and useful comments. EGD was partially funded by grant ECO2013–47092 (Ministerio de Economía y Competitividad, Spain).

References

Akaike, H. (1974). A new look at the statistical model identification. *IEEE Transactions on Automatic Control*, 19(6):716–723.

Anders, A. and Rotthoff, K. (2011). Yellow cards: Do they matter? *Journal of Quantitative Analysis in Sports*, 7(1):1–12.

Boyko, R., Boyko, A., and Boyko, M. (2007). Referee bias contributes to home advantage in english premiership football. *Journal of Sports Sciences*, 25:1185–1194.

Cacaoullos, T. and Papageorgiou, P. (1982). Bivariate negative binomial-Poisson and negative binomial-Bernoulli models with an applicaton to accident data. *G. Kallianpur, P.R. Krishnaiah, J.K. Ghosh, eds. Statistics and Probability Essays in Honor of C.R. Rao*, pages 155–168.

Cameron, C. and Trivedi, P. (1998). *Regression Analysis of Count Data.* Cambridge University Press.

Consul, P. C. (1990). On some properties and applications of quasi-binomial distribution. *Communications in Statistics-Theory and Methods*, 19(2):477–504.

Dawson, P. (2012). Experience, social pressure and preformance: the case of soccer officials. *Applied Economic Letters*, 19:883–886.

Dawson, P., Dobson, S., Goddard, J., and Wilson, J. (2007). Are football referees really biased and inconsistent? evidence from the english premier league. *Journal of the Royal Statistical Society, Series A*, 170:231–250.

Dohmen, T. (2008). The influence of social forces: Evidence from the behavior of football referees. *Economic Inquiry*, 46:411–424.

Greenhough, J., Birch, P., Chapman, S., and Rowlands, G. (2002). Football goal distributions and extremal statistics. *Physica A*, 316:615–624.

Kim, D. (2013). A simple zero bivariate negative binomial regression model with different dispersion parameters. *Journal of the Korean Data & Information Science Society*, 24(4):895–900.

Leiter, R. and Hamdan, M. (1973). Some bivariate probability models applicable to traffic accidents and fatalities. *Int. Stat. Rev.*, 41(1):87–100.

Lex, H., Pizzera, A., Kurtes, M., and Schack, T. (2015). Influence of players' vocalisations on soccer referees' decisions. *European Journal of Sport Science*, 15(4):424–428.

Nevill, A., Balmer, N., and Williams, A. (2002). The influence of crowd noise and experience upon refereeing decisions in football. *Psychology of Sport and Exercise*, 3:261–272.

Reilly, B. and Witt, R. (2013). Red cards, referee home bias and social pressure: evidence from English premiership soccer. *Applied Economic Letters*, 20(7):710–714.

Ridder, G., Cramer, J., and Hopstaken, P. (1994). Down to ten: Estimating the effect of a red card in soccer. *Journal of the American Statistical Association*, 89(427):1124–1127.

Vecer, J., Kopriva, F., and Ichiba, T. (2009). Estimating the effect of the red cards in soccer: When to commit an ofense in exchange for preventing a goal opportunity. *Journal of Quantitative Analysis in Sports*, 18(1):1–20.

Wolfram, S. (2003). *The Mathematica Book*. Wolfram Media, Inc.

Appendix

In this section, we derive estimators based on maximum likelihood for the model with and without covariates, and also provide closed–form expressions for the Fisher information matrix.

Estimating the model parameters without covariates

Here we present the estimation based on the maximum likelihood method, first considering the model without covariates. Let $\Theta = (\mu_1, \mu_2)$ and a sample consisting of n observations $(\tilde{x}, \tilde{z}) = \{(x_1, z_1), \ldots, (x_n, z_n)\}$, taken from the probability function (5). The log–likelihood is proportional to

$$\ell(\Theta; \tilde{x}, \tilde{z}) \propto n\bar{z} \log \mu_2 + n(\bar{x} - \bar{z}) \log(\mu_1 - \mu_2) - n\mu_1,$$

where $\bar{x}$ and $\bar{z}$ are the sample means of X and Z, respectively. The normal equations to be solved are

$$\begin{aligned}
\frac{\partial \ell(\Theta; \tilde{x}, \tilde{z})}{\partial \mu_1} &= \frac{n(\bar{x} - \bar{z})}{\mu_1 - \mu_2} - n = 0, \\
\frac{\partial \ell(\Theta; \tilde{x}, \tilde{z})}{\partial \mu_2} &= \frac{n\bar{z}}{\mu_2} + \frac{n(\bar{z} - \bar{x})}{\mu_1 - \mu_2} = 0,
\end{aligned}$$

from which it is easy to obtain the solution of this system and to see that the maximum likelihood estimators are $\widehat{\mu}_1 = \bar{x}$ and $\widehat{\mu}_2 = \bar{z}$. The second partial derivatives are

$$\begin{aligned}
\frac{\partial^2 \ell(\Theta; \tilde{x}, \tilde{z})}{\partial \mu_1^2} &= -\frac{n(\bar{x} - \bar{z})}{(\mu_1 - \mu_2)^2}, \\
\frac{\partial^2 \ell(\Theta; \tilde{x}, \tilde{z})}{\partial \mu_2^2} &= -\frac{n\bar{z}}{\mu_2^2} + \frac{n(\bar{z} - \bar{x})}{(\mu_1 - \mu_2)^2}, \\
\frac{\partial^2 \ell(\Theta; \tilde{x}, \tilde{z})}{\partial \mu_1 \partial \mu_2} &= \frac{n(\bar{x} - \bar{z})}{(\mu_1 - \mu_2)^2}.
\end{aligned}$$

The expectation of the second partial derivative, with a sign change, yields the Fisher information matrix, and so this matrix is obtained as follows: $\mathcal{J}(\widehat{\Theta}) = -E\left[\frac{\partial^2}{\partial \widehat{\Theta}^2} \ell(\widehat{\Theta}; \tilde{x}, \tilde{z})\right]$, which is given by

$$\mathcal{J}(\widehat{\Theta}) = \begin{bmatrix} \frac{n}{\widehat{\mu}_1 - \widehat{\mu}_2} & \frac{n\widehat{\mu}_1}{\widehat{\mu}_2(\widehat{\mu}_1 - \widehat{\mu}_2)} \\ \frac{n\widehat{\mu}_1}{\widehat{\mu}_2(\widehat{\mu}_1 - \widehat{\mu}_2)} & \frac{n}{\widehat{\mu}_2 - \widehat{\mu}_1} \end{bmatrix}.$$

The asymptotic variance–covariance matrix of $(\widehat{\mu}_1, \widehat{\mu}_2)$ is obtained by inverting this information matrix.

Estimating the model parameters with covariates

When covariates are considered, the log–likelihood is now proportional to

$$\ell(\beta_1,\beta_2;\tilde{x},\tilde{z}) \propto \sum_{i=1}^{n} \left[z_i \log \mu_{2i} + (x_i - z_i)\log(\mu_{1i}-\mu_{2i}) - \mu_{1i}\right]. \tag{6}$$

Observe now that that $\mu_{1i} = \mu_{1i}(\beta_1)$ while $\mu_{2i} = \mu_{2i}(\beta_1, \beta_2)$, denoting that the first depends only on β_1 and the second on both, β_1 and β_2. Thus,

$$\begin{aligned}
\frac{\partial \mu_{1i}}{\partial \beta_{1j}} &= \omega_{ji}\mu_{1i},\\
\frac{\partial \mu_{2i}}{\partial \beta_{1j}} &= \omega_{ji}\mu_{2i},\\
\frac{\partial \mu_{2i}}{\partial \beta_{2j}} &= \frac{\mu_{2i}\eta_{ji}}{1+\exp(\eta_{2i})},
\end{aligned}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Then, after some algebra we obtain the normal equations, which are given by

$$\begin{aligned}
\frac{\partial \ell(\beta_1,\beta_2;\tilde{x},\tilde{z})}{\partial \beta_{1j}} &= \sum_{i=1}^{n} \omega_{ji}(x_i - \mu_{1i}) = 0, \quad j = 1,\ldots,m,\\
\frac{\partial \ell(\beta_1,\beta_2;\tilde{x},\tilde{z})}{\partial \beta_{2j}} &= \sum_{i=1}^{n} \frac{\eta_{ji}\phi(\mu_{1i},\mu_{2i},x_i,z_i)}{1+\exp(\eta_{2i}\beta_2)} = 0, \quad j = 1,\ldots,m,
\end{aligned}$$

where

$$\phi(\mu_{1i},\mu_{2i},x_i,z_i) = \frac{z_i\mu_{1i} - x_i\mu_{2i}}{\mu_{1i}-\mu_{2i}}.$$

These equations provide the maximum likelihood estimates $\widehat{\beta}_1 = (\widehat{\beta}_{11}, \ldots, \widehat{\beta}_{1m})$ and $\widehat{\beta}_2 = (\widehat{\beta}_{21}, \ldots, \widehat{\beta}_{2m})$.

The second partial derivatives are as follows.

$$\begin{aligned}
\frac{\partial^2 \ell(\beta_1,\beta_2;\tilde{x},\tilde{z})}{\partial \beta_{1j}^2} &= -\sum_{i=1}^{n} \omega_{ji}^2 \mu_{1i}, \quad j=1,\ldots,m,\\
\frac{\partial^2 \ell(\beta_1,\beta_2;\tilde{x},\tilde{z})}{\partial \beta_{1j}\partial \beta_{1k}} &= -\sum_{i=1}^{n} \omega_{ji}\omega_{ki} \mu_{1i}, \quad j\neq k,\\
\frac{\partial^2 \ell(\beta_1,\beta_2;\tilde{x},\tilde{z})}{\partial \beta_{1j}\beta_{2j}} &= 0, \quad j=1,\ldots,m,\\
\frac{\partial^2 \ell(\beta_1,\beta_2;\tilde{x},\tilde{z})}{\partial \beta_{2j}^2} &= -\sum_{i=1}^{n} \left(\frac{\eta_{ji}}{1+\exp(\eta_{2i}\beta_2)}\right)^2 \left[\phi(\mu_{1i},\mu_{2i},x_i,z_i)\exp(\eta_{2i}\beta_2)\right.\\
&\quad \left. + \frac{(x_i-\phi(\mu_{1i},\mu_{2i},x_i,z_i))\mu_{2i}}{\mu_{1i}-\mu_{2i}}\right], \quad j=1,\ldots,m.\\
\frac{\partial^2 \ell(\beta_1,\beta_2;\tilde{x},\tilde{z})}{\partial \beta_{2j}\partial \beta_{2k}} &= -\sum_{i=1}^{n} \frac{\eta_{ji}\eta_{ki}}{(1+\exp(\eta_{2i}\beta_2))^2} \left[\phi(\mu_{1i},\mu_{2i},x_i,z_i)\exp(\eta_{2i}\beta_2)\right.\\
&\quad \left. + \frac{(x_i-\phi(\mu_{1i},\mu_{2i},x_i,z_i))\mu_{2i}}{\mu_{1i}-\mu_{2i}}\right], \quad j=1,\ldots,m.
\end{aligned}$$

Now, the elements of Fisher's information matrix (with dimension $m\times m$) are given by

$$\begin{aligned}
E\left(-\frac{\partial \ell(\widehat{\beta}_1,\widehat{\beta}_2;\tilde{x},\tilde{z})}{\partial \widehat{\beta}_{1j}^2}\right) &= \sum_{i=1}^{n} \omega_{ji}^2 \widehat{\mu}_{1i},\\
E\left(-\frac{\partial^2 \ell(\widehat{\beta}_1,\widehat{\beta}_2;\tilde{x},\tilde{z})}{\partial \widehat{\beta}_{1j}\partial \widehat{\beta}_{1k}}\right) &= \sum_{i=1}^{n} \omega_{ji}\omega_{ki} \widehat{\mu}_{1i}, \quad j\neq k,\\
E\left(-\frac{\partial \ell(\widehat{\beta}_1,\widehat{\beta}_2;\tilde{x},\tilde{z})}{\partial \widehat{\beta}_{1j}\partial \widehat{\beta}_{2j}}\right) &= 0,\\
E\left(-\frac{\partial \ell(\widehat{\beta}_1,\widehat{\beta}_2;\tilde{x},\tilde{z})}{\partial \beta_{2j}^2}\right) &= \sum_{i=1}^{n} \frac{\widehat{\mu}_{1i}\widehat{\mu}_{2i}}{\widehat{\mu}_{1i}-\widehat{\mu}_{2i}} \left(\frac{\eta_{ji}}{1+\exp(\eta_{2i}\widehat{\beta}_2)}\right)^2,\\
E\left(-\frac{\partial \ell(\widehat{\beta}_1,\widehat{\beta}_2;\tilde{x},\tilde{z})}{\partial \beta_{2j}\partial \beta_{2k}}\right) &= \sum_{i=1}^{n} \frac{\widehat{\mu}_{1i}\widehat{\mu}_{2i}}{\widehat{\mu}_{1i}-\widehat{\mu}_{2i}} \frac{\eta_{ji}\eta_{ki}}{(1+\exp(\eta_{2i}\widehat{\beta}_2))^2}, \quad j\neq k,
\end{aligned}$$

for $j=1,\ldots,m$, where we have taken into account that $E(\phi(\mu_{1i},\mu_{2i},x_i,z_i))=0$. Again, the asymptotic variance–covariance matrix of $(\widehat{\beta}_1,\widehat{\beta}_2)$ is obtained by inverting this information matrix.