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On the finite mixture of exponential, Rayleigh and Burr Type-XII distributions: estimation of parameters in Bayesian framework

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In recent years, the finite mixtures of distributions have been proved to be of considerable attention in terms of their practical applications. This paper focuses on studying the problem of estimating the parameters of a 3-component mixture of exponential, Rayleigh and Burr Type-XII distributions using Type-I right censoring scheme in Bayesian framework. The expressions for the Bayes estimators and their variances using the non-informative and the informative priors are derived for censored sample as well as for complete sample. The hyperparameters are elicited using prior predictive distribution. The posterior predictive distribution with different priors is derived and the equations necessary to find the lower and upper limits of the Bayesian predictive intervals are constructed. A detailed simulation study is carried out to investigate the performance (in terms of variances) of the Bayes estimators. Finally, the model is illustrated using the real life data. Bayes estimators using the informative prior have been observed performing superior.

Keywords: Mixture distribution; posterior distribution; Bayes estimator; posterior risk; elicitation; predictive interval.

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1 Introduction

The inspiration towards the use of the Bayesian mixture model is that it allows rotting the complex structure of mixture models to the simple models when the number of components is unknown. Li (1983) and Li et al. (1988) spotlight several features of mixture models and explain two types of mixture models. If the component distributions of a mixture belong to same family, their mixture is known as a Type-I mixture model. A Type-II mixture model is defined if the component distributions of a mixture belong to different families. In practical situations, we need to infer only about the mixing proportions when a mixture population may have known component densities. Also, in many real-life applications, there are known component densities with unknown parameters but mixing proportions are known and the vice versa. In this paper, Type-II mixture models with unknown parameters of the known number of component densities belonging to the different parametric family having unknown mixing proportions are considered.

Mixture models have been paid great attention in in terms of their methodological development and practical areas. The use of the finite mixture model becomes unavoidable when the data are given only from overall mixture distributions rather not given for each component. Modeling these data as a mixture of some component distributions is known as direct application of the mixture models. The direct applications of mixture models can be seen mostly in engineering, medical sciences, biological sciences, agriculture, economics, life testing, reliability and survival analysis etc. Due to rapid growth of powerful computational techniques in recent few years, the number of applied fields where the mixture models have been used is still extending. In many applications, available data can be considered as coming from a mixture of two or more distributions. For example, samples of sand are often analyzed by measuring the frequency distribution of grain sizes. The sand may be known to be a mixture of several minerals. It is of interest to estimate the proportions of different minerals in the sand. It may also be desired to estimate the grain size distributions for the different minerals. If the component densities of the different minerals are known to be belonging to different families, the mixture distribution is called Type-II mixture distribution. In our study, we took the same situation where the number of minerals is three and distribution of each mineral is different (e.g. exponential, Rayleigh and Burr Type- XII). This idea enables us to mix statistical distributions to get a new mixture distribution. Using this idea, a finite mixture of some suitable probability distribution is suggested to study a population that is supposed to comprise a number of subpopulations mixed in unknown proportions. In this paper, a population of certain objects is assumed to be composed of three subgroups mixed together in two unknown mixing proportions. The random observations taken from this population are supposed to be characterized by one of the three distinct unknown members of distributions. So the 3-component mixture distribution is recommended to model this population.

Several authors have extensively applied the 2- component mixture modeling in different practical problems using Bayesian analysis. For a detailed review of Bayesian estimation techniques, discussion and applications of mixture modeling, one can refer to

Liu et al. (2010), Saleem and Irfan (2010), AL-Hussaini and Hussein (2012), Ali et al. (2012), Mohammadi and Salehi-Rad (2012), Abd and AL-Zaydi (2013), Ali et al. (2012), Feroze and Aslam (2014), Mohamed et al. (2014), Zhang and Huang (2015), Tian et al. (2014) and Feroze (2015). Plenty of work on mixture distributions from the Bayesian point of view motivated us to establish the popularity of Bayesian inference using mixture distribution and explore it further in case of Type-II mixture distributions arising from different fields of interest such as survival and reliability engineering.

Several types of data are used in daily life, including simple data, grouped data, censored data, progressively censored data and record values. The idea to use the censoring scheme is that censoring is a property of datasets and not of parameters and is an unavoidable feature of the lifetime applications. Censoring is an important and valuable aspect of the lifetime data. Due to time and cost problem, it is impossible to continue the testing until the last observation in order to obtain a complete data set. An account of censoring can be seen in Gijbels (2010) and Kalbfleisch and Prentice (2011). In this study, an ordinary Type-I right censoring scheme is considered and the observations greater than the fixed cut off censoring value (the number of dead objects) are taken as censored ones.

Motivated by the applications of mixture models in a wide range of applied fields, in current study, specifically, we plan to have Bayesian analysis of a 3-component mixture of exponential, Rayleigh and Burr Type-XII distributions. Also, in this study, the direct application of mixture model is considered under Type-II mixture modeling.

The rest of the article is organized as follows: The development of a 3-component mixture distribution is given in Section 2. The likelihood function of a 3-component mixture of exponential, Rayleigh and Burr Type-XII distributions are defined in Section 3. The joint posterior distributions assuming the non-informative and the informative priors are derived in Section 4. In Section 5, the Bayes estimators and their variances are derived. The posterior predictive distribution and the Bayesian predictive intervals are described in Section 6. The elicitation of hyperparameters is discussed in Section 7. The limiting expressions for complete data set are constructed in Section 8. The simulation study and real life data example are explained in Sections 9 and 10, respectively. Finally, the conclusion of this study is given in Section 11.

2 The 3-Component Mixture Model

A finite 3-component mixture distribution with unknown mixing proportions p_1, p_2 and $(1-p_1-p_2)$ has its probability density function (pdf) as:

$$f(y) = p_1 f_1(y) + p_2 f_2(y) + (1-p_1-p_2) f_3(y), p_1,\ p_2 \geq 0,\ p_1+p_2 \leq 1, \tag{1}$$

So the finite 3-component mixture distribution (1) assuming exponential, Rayleigh and Burr Type-XII distributions for 1^{st}, 2^{nd} and 3^{rd} components with unknown parameters λ_1, λ_2 and λ_3, respectively, has the following form of pdf:

$$f(y) = p_1\lambda_1 \exp(-\lambda_1 y) + p_2 \frac{y}{\lambda_2^2} \exp\left(-\frac{y^2}{2\lambda_2^2}\right) + (1-p_1-p_2)\lambda_3 (y+1)^{-(\lambda_3+1)}. \tag{2}$$

The corresponding a 3-component mixture cumulative distribution function (cdf) is:

$$F(y) = p_1 F_1(y_1) + p_2 F_2(y) + (1 - p_1 - p_2) F_3(y),$$

$$F(y) = 1 - p_1 \exp(-\lambda_1 y) - p_2 \exp\left(-\frac{y^2}{2\lambda_2^2}\right) - (1 - p_1 - p_2)(y+1)^{-(\lambda_3+1)}.$$

3 The Likelihood Function

From a 3-component mixture of exponential, Rayleigh and Burr Type-XII distributions, let n units are employed in a life testing experiment with fixed test termination time t. Now, the experiment be performed and it is detected that r units out of n units terminated until fixed test termination time t and the remaining $n-r$ units are still working. In many real-life problems, only the failed objects can easily be pointed out as members of either subpopulation-1 or subpopulation-2 or subpopulation-3 which is consistent with the definition of Type-I right censoring scheme. It is to be noted that out of r failures, r_1, r_2 and r_3 failures can be categorized as belong to subpopulation-1, subpopulation-2 and subpopulation-3, respectively, depending upon the reason of failure. So, the numbers of uncensored observations is $r = r_1 + r_2 + r_3$. The remaining $n-r$ observations are the censored observations that provide no information about the subpopulation to which they belong. Now, we define y_{lk}, $0 < y_{lk} \le t$, be the failure time of the k^{th} unit belonging to the l^{th} subpopulation, where $l = 1,\ 2,\ 3$ and $k = 1,\ 2, \cdots,\ r_l$.

The likelihood function for a finite 3-component mixture distribution is:

$$\begin{aligned} L(\lambda_1, \lambda_2, \lambda_3, p_1, p_2|\mathbf{y}) \propto & \left\{\prod_{k=1}^{r_1} p_1 f_1(y_{1k})\right\}\left\{\prod_{k=1}^{r_2} p_2 f_2(y_{2k})\right\} \\ \times & \left\{\prod_{k=1}^{r_3} (1-p_1-p_2) f_3(y_{3k})\right\}\{1 - F(t)\}^{n-r}. \end{aligned}$$

After substitution and simplification, the likelihood function of a 3-component mixture of exponential, Rayleigh and Burr Type-XII distributions is:

$$\begin{aligned} L(\lambda_1, \lambda_2, \lambda_3, p_1, p_2|\mathbf{y}) \propto & \left[\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp\left\{-\left((n-r-i)t + \sum_{k=1}^{r_1} y_{1k}\right)\lambda_1\right\}\right. \\ \times & \exp\left\{-\left((i-j)\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_2} y_{2k}^2\right)\lambda_2^{-2}\right\} \\ \times & \exp\left\{-\left(j\ln(t+1) + \sum_{k=1}^{r_1}\ln(y_{3k}+1)\right)\lambda_3\right\} \\ \times & \left.\lambda_1^{r_1}\lambda_2^{-2r_2}\lambda_3^{r_3} p_1^{n-r-i+r_1} p_2^{i-j+r_2}(1-p_1-p_2)^{j+r_3}\right], \end{aligned}$$

where $\mathbf{y} = (y_{11},\ y_{12}, \ldots,\ y_{1r_1},\ y_{21},\ y_{22}, \ldots,\ y_{2r_2},\ y_{31},\ y_{32}, \ldots,\ y_{3r_3})$ are the observed failure times for the uncensored observations.

4 The Joint Posterior Distribution

In this section, the joint posterior distributions of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 given data, say $\mathbf{y}$, are derived using the non-informative (Jeffreys') and the informative priors.

4.1 The Joint Posterior Distribution using the Non-informative Prior

There exist situations where no prior information on the parameter of interest is available. In such situations, one has to use a non-informative prior (NIP) distribution. The most commonly used the NIP is the Jeffreys' prior (JP) when no formal prior information is available. According to Jeffreys (1946), the JP for $\lambda_1,\ \lambda_2$ and λ_3 is defined as $p(\lambda_m) \propto \sqrt{|I(\lambda_m)|}$, where $m = 1,\ 2,\ 3$ and $I(\lambda_m) = -E\left\{\frac{\partial^2 f(y;\lambda_m)}{\partial\lambda_m^2}\right\}$ is Fisher's information. It is interesting to note that the JP for proportion parameters p_1 and p_2 cannot be assumed under the current settings. Therefore, the uniform distribution over the interval (0, 1) is assumed as the NIP for both proportion parameters p_1 and p_2 [i.e., $p_1 \sim (0,\ 1)$ and $p_2 \sim (0,\ 1)$]. Under the assumption of independence of all the parameters, the joint prior distribution of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 is:

$$\xi_1(\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2) \propto \frac{1}{\lambda_1\lambda_2\lambda_3}, \lambda_1,\ \lambda_2,\ \lambda_3 > 0,\ p_1,\ p_2 \geq 0,\ p_1 + p_2 \leq 1. \quad (3)$$

The joint posterior distribution of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 given data $\mathbf{y}$ using the JP is:

$$\begin{aligned} &\pi_1(\lambda_1, \lambda_2, \lambda_3, p_1, p_2|\mathbf{y}) \\ = &\frac{L(\lambda_1, \lambda_2, \lambda_3, p_1, p_2|y)\,\xi_1(\lambda_1, \lambda_2, \lambda_3, p_1, p_2)}{\int\limits_{p_2}\int\limits_{p_1}\int\limits_{\lambda_3}\int\limits_{\lambda_2}\int\limits_{\lambda_1} L(\lambda_1, \lambda_2, \lambda_3, p_1, p_2|y)\,\xi_1(\lambda_1, \lambda_2, \lambda_3, p_1, p_2)\, d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2} \end{aligned}$$

$$\begin{aligned} \pi_1(\lambda_1, \lambda_2, \lambda_3, p_1, p_2|\mathbf{y}) &= \sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\exp(-D_{11}\lambda_1)\exp\left(-D_{21}\lambda_2^{-2}\right) \\ &\times \frac{\exp(-D_{31}\lambda_3)\, p_1^{A_{01}-1} p_2^{B_{01}-1}(1-p_1-p_2)^{C_{01}-1}}{\Omega_1\lambda_1^{1-A_{11}}\lambda_2^{2A_{21}+1}\lambda_3^{1-A_{31}}} \end{aligned} \quad (4)$$

where $A_{11} = r_1,\ A_{21} = r_2,\ A_{31} = r_3,\ A_{01} = n - r - i + r_1 + 1,\ B_{01} = i - j + r_2 + 1,$ $C_{01} = j + r_3 + 1,$

$$D_{11} = (n - r - i)\, t + \sum_{k=1}^{r_1} y_{1k}, D_{21} = (i - j)\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_2} y_{2k}^2, D_{31} = j\ln(t+1) + \sum_{k=1}^{r_3}\ln(y_{3k}+1),$$

$$\Omega_1 = \frac{1}{2}\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix} B(A_{01}, B_{01}, C_{01})\, D_{11}^{-A_{11}} D_{21}^{-A_{21}} D_{31}^{-A_{31}}.$$

The marginal posterior distribution of each parameter is obtained by integrating out the remaining four parameters.

4.2 The Joint Posterior Distribution using the Informative Prior

When definite information is available, it is quantified as an informative prior (IP). Using an informative prior along with the sample information is usually thought of as updating the current information which, in result, helps reducing the variances of the Bayes estimators.Now, we assume the gamma, square root inverted gamma (SRIG) and gamma distributions as IP for component parameters $\lambda_1,\ \lambda_2$ and λ_3, respectively, [i.e.,$\lambda_1 \sim Gamma\,(a_1,\ b_1)$, $\lambda_2 \sim SRIG\,(a_2,\ b_2)$ and $\lambda_3 \sim Gamma\,(a_3,\ b_3)$] and a bivariate beta distribution as IP for proportion parameters p_1 and p_2 [i.e., $p_1,\ p_2 \sim Bivariate\ Beta\,(a,\ b,\ c)$]. So, assuming the independence of parameters, the joint prior distribution of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 is:

$$\xi_2\left(\lambda_1,\lambda_2,\lambda_3,p_1,p_2\right) \propto \left.\begin{array}{l} \lambda_1^{a_1-1}\exp\left(-b_1\lambda_1\right)\lambda_2^{-(2a_2+1)}\exp\left(-b_2\lambda_2^{-2}\right)\lambda_3^{a_3-1} \\ \times\exp\left(-b_3\lambda_3\right)p_1^{a-1}p_2^{b-1}(1-p_1-p_2)^{c-1} \end{array}\right\} \tag{5}$$

In this case, the joint posterior distribution of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 given data $\mathbf{y}$ using the IP is:

$$\pi_2\left(\lambda_1,\lambda_2,\lambda_3,p_1,p_2|\mathbf{y}\right) = \left.\begin{array}{l} \sum\limits_{i=0}^{n-r}\sum\limits_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\exp\left(-D_{12}\lambda_1\right)\exp\left(-D_{22}\lambda_2^{-2}\right) \\ \times\frac{\exp(-D_{32}\lambda_3)p_1^{A_{02}-1}p_2^{B_{02}-1}(1-p_1-p_2)^{C_{02}-1}}{\Omega_2\lambda_1^{1-A_{12}}\lambda_2^{2A_{22}+1}\lambda_3^{1-A_{32}}} \end{array}\right\} \tag{6}$$

where $A_{12} = r_1+a_1$, $A_{22} = r_2+a_2$, $A_{32} = r_3+a_3$, $A_{02} = n-r-i+r_1+a$, $B_{02} = i-j+r_2+b$, $C_{02} = j + r_3 + c$, $D_{12} = (n-r-i)\,t + \sum_{k=1}^{r_1} y_{1k} + b_1$, $D_{22} = (i-j)\,\frac{t^2}{2} + \frac{1}{2}\sum_{k=1}^{r_2} y_{2k}^2 + b_2$, $D_{32} = j\ln(t+1) + \sum_{k=1}^{r_3}\ln(y_{3k}+1) + b_3$,

$$\Omega_2 = \frac{1}{2}\Gamma\left(A_{12}\right)\Gamma\left(A_{22}\right)\Gamma\left(A_{32}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}B\left(A_{02},B_{02},C_{02}\right)D_{12}^{-A_{12}}D_{22}^{-A_{22}}D_{32}^{-A_{32}}.$$

For each parameter, the marginal posterior distribution is obtained by integrating out the remaining four parameters.

5 Bayes Estimators and Variances

The expectation of each parameter with their respective marginal posterior distribution gives the Bayes estimator of the parameter. So, the expressions for the Bayes estimators of the parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 using the NIP (JP) and the IP are obtained as:

$$\hat{\lambda}_1|\mathbf{y} = \left.\begin{array}{l} \frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v} \\ \times\sum\limits_{i=0}^{n-r}\sum\limits_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{(A_{1v}+1)}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}} \end{array}\right\} \tag{7}$$

$$\hat{\lambda}_2|\mathbf{y} = \left.\begin{array}{l} \frac{\Gamma(A_{1v})\Gamma\left(A_{2v}-\frac{1}{2}\right)\Gamma(A_{3v})}{2\Omega_v} \\ \times\sum\limits_{i=0}^{n-r}\sum\limits_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{\left(A_{2v}-\frac{1}{2}\right)}D_{3v}^{A_{3v}}} \end{array}\right\} \tag{8}$$

$$\hat{\lambda}_3|\mathbf{y} = \left.\begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{2\Omega_v} \\ \times \sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{(A_{3v}+1)}} \end{array}\right\} \tag{9}$$

$$\hat{p}_1|\mathbf{y} = \left.\begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v} \\ \sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}} \end{array}\right\} \tag{10}$$

$$\hat{p}_2|\mathbf{y} = \left.\begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v} \\ \times \sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}} \end{array}\right\} \tag{11}$$

When presenting the Bayes estimators, it is usually necessary to specify the accuracy of the Bayes estimators. The customary Bayesian measure of the accuracy of a Bayes estimator is the variance of the Bayes estimator under squared error loss function. So, the variances of the Bayes estimators of the parameters λ_1, λ_2, λ_3, p_1 and p_2 using the NIP (JP) and the IP are derived as:

$$\begin{aligned} Var\left(\hat{\lambda}_1|\mathbf{y}\right) &= \frac{\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{(A_{1v}+2)}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}} \\ &- \left\{\frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{(A_{1v}+1)}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}}\right\}^2 \end{aligned} \tag{12}$$

$$\begin{aligned} Var\left(\hat{\lambda}_2|\mathbf{y}\right) &= \frac{\Gamma(A_{1v})\Gamma(A_{2v}-1)\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{(A_{2v}-1)}D_{3v}^{A_{3v}}} \\ &- \left\{\frac{\Gamma(A_{1v})\Gamma\left(A_{2v}-\frac{1}{2}\right)\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{\left(A_{2v}-\frac{1}{2}\right)}D_{3v}^{A_{3v}}}\right\}^2 \end{aligned} \tag{13}$$

$$\begin{aligned} Var\left(\hat{\lambda}_3|\mathbf{y}\right) &= \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{(A_{3v}+2)}} \\ &- \left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{(A_{3v}+1)}}\right\}^2 \end{aligned} \tag{14}$$

$$\begin{aligned} Var\left(\hat{p}_1|\mathbf{y}\right) &= \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}} \\ &- \left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}}\right\}^2 \end{aligned} \tag{15}$$

$$Var\left(\hat{p}_2|\mathbf{y}\right)=\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix}n-r\\ i\end{pmatrix}\begin{pmatrix}i\\ j\end{pmatrix}\frac{B(A_{0v},\,C_{0v})B(B_{0v}+2,\,A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}}$$
$$-\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix}n-r\\ i\end{pmatrix}\begin{pmatrix}i\\ j\end{pmatrix}\frac{B(A_{0v},\,C_{0v})B(B_{0v}+1,\,A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}}\right\}^2. \tag{16}$$

6 Elicitation of Hyperparameters

Elicitation is a method used to formulate a person's belief and knowledge about one or more uncertain quantities into a (joint) probability density function for those quantities. In Bayesian perspective, elicitation can be regarded as a technique to quantification of prior knowledge about the random parameter(s) so that this can then be combined with the likelihood to obtain posterior distribution for further statistical analysis. Authors who have discussed this problem include Kadane et al. (1980), Gavasakar (1988), Al-Awadhi and Garthwaite (2001), Aslam (2003), Hahn (2006) and Saleem and Aslam (2009). In this study, we adopted prior predictive method based on predictive probabilities given by Aslam (2003).

For eliciting the hyperparameters, the prior predictive distribution for a random variable X using the IP is:

$$p\left(x\right)=\int_{p_2}\int_{p_1}\int_{\lambda_3}\int_{\lambda_2}\int_{\lambda_1} f\left(x|\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2\right)\xi_2\left(\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2\right)d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2. \tag{17}$$

On substituting (2) and (5) in (17) and then simplifying, we get:

$$p\left(x\right)=\frac{1}{\left(a+b+c\right)}\left(\frac{aa_1b_1^{a_1}}{\left(b_1+x\right)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{\left(x+1\right)}\frac{ca_3b_3^{a_3}}{\left\{b_3+\ln\left(x+1\right)\right\}^{a_3+1}}\right). \tag{18}$$

Using the prior predictive distribution given in (18), we consider nine intervals $0\le x\le 0.5$, $0.5\le x\le 1$, $1\le x\le 1.5$, $1.5\le x\le 2$, $2\le x\le 2.5$, $2.5\le x\le 3$, $3\le x\le 3.5$, $3.5\le x\le 4$ and $4\le x\le 4.5$ with respective probabilities 0.30, 0.20, 0.15, 0.10, 0.06, 0.04, 0.03, 0.02 and 0.01. It is worth mentioning that these probabilities might have been obtained from the expert(s) as their opinion about the likelihood of these intervals. Moreover, different intervals could also be considered. Using (18), following nine equations in (19)-(27) are solved simultaneously in Mathematica package for eliciting the hyperparameters $a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3,\ a,\ b$ and c.

$$\frac{1}{\left(a+b+c\right)}\int_0^{0.5}\left(\frac{aa_1b_1^{a_1}}{\left(b_1+x\right)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{\left(x+1\right)}\frac{ca_3b_3^{a_3}}{\left\{b_3+\ln\left(x+1\right)\right\}^{a_3+1}}\right)dx=0.30 \tag{19}$$

$$\frac{1}{\left(a+b+c\right)}\int_{0.5}^{1}\left(\frac{aa_1b_1^{a_1}}{\left(b_1+x\right)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{\left(x+1\right)}\frac{ca_3b_3^{a_3}}{\left\{b_3+\ln\left(x+1\right)\right\}^{a_3+1}}\right)dx=0.20 \tag{20}$$

$$\frac{1}{(a+b+c)}\int_{1}^{1.5}\left(\frac{aa_1b_1^{a_1}}{(b_1+x)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{(x+1)}\frac{ca_3b_3^{a_3}}{\{b_3+\ln(x+1)\}^{a_3+1}}\right)dx=0.15 \tag{21}$$

$$\frac{1}{(a+b+c)}\int_{1.5}^{2}\left(\frac{aa_1b_1^{a_1}}{(b_1+x)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{(x+1)}\frac{ca_3b_3^{a_3}}{\{b_3+\ln(x+1)\}^{a_3+1}}\right)dx=0.10 \tag{22}$$

$$\frac{1}{(a+b+c)}\int_{2}^{2.5}\left(\frac{aa_1b_1^{a_1}}{(b_1+x)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{(x+1)}\frac{ca_3b_3^{a_3}}{\{b_3+\ln(x+1)\}^{a_3+1}}\right)dx=0.06 \tag{23}$$

$$\frac{1}{(a+b+c)}\int_{2.5}^{3}\left(\frac{aa_1b_1^{a_1}}{(b_1+x)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{(x+1)}\frac{ca_3b_3^{a_3}}{\{b_3+\ln(x+1)\}^{a_3+1}}\right)dx=0.04 \tag{24}$$

$$\frac{1}{(a+b+c)}\int_{3}^{3.5}\left(\frac{aa_1b_1^{a_1}}{(b_1+x)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{(x+1)}\frac{ca_3b_3^{a_3}}{\{b_3+\ln(x+1)\}^{a_3+1}}\right)dx=0.03 \tag{25}$$

$$\frac{1}{(a+b+c)}\int_{3.5}^{4}\left(\frac{aa_1b_1^{a_1}}{(b_1+x)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{(x+1)}\frac{ca_3b_3^{a_3}}{\{b_3+\ln(x+1)\}^{a_3+1}}\right)dx=0.02 \tag{26}$$

$$\frac{1}{(a+b+c)}\int_{4}^{4.5}\left(\frac{aa_1b_1^{a_1}}{(b_1+x)^{a_1+1}}+x\frac{ba_2b_2^{a_2}}{\left(b_2+x^2/2\right)^{a_2+1}}+\frac{1}{(x+1)}\frac{ca_3b_3^{a_3}}{\{b_3+\ln(x+1)\}^{a_3+1}}\right)dx=0.01. \tag{27}$$

The elicited values of the hyperparameters $a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3,\ a,\ b$ and c are obtained as 2.7944, 2.4352, 0.67532, 0.61495, 1.9490, 1.6535, 1.0939, 1.3719 and 1.0347, respectively.

7 The Posterior Predictive Distribution and Bayesian Predictive Interval

In this section, we present the derivation of posterior predictive distribution using the JP and the IP.

7.1 The Posterior Predictive Distribution

The posterior predictive distribution contains the information about the future observation $Z = Y_{n+1}$ of a random variable given the data $\mathbf{y}$, already observed. Arnold and Press (1983), Al-Hussaini et al. (2001), Al-Hussaini and Ahmad (2003), and Bolstad (2004) have given a detailed discussion on prediction and predictive distribution under the Bayesian paradigm.

The posterior predictive distribution of a future observation $Z = Y_{n+1}$ given data $\mathbf{y}$ using the JP and the IP is:

$$f(z\,|\mathbf{y}) = \int_{p_2}\int_{p_1}\int_{\lambda_3}\int_{\lambda_2}\int_{\lambda_1} f(z|\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2)\,\pi_v(\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2|\mathbf{y})\ d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2, \tag{28}$$

where $f(z|\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2) = p_1 f_1(z|\lambda_1) + p_2 f_2(z|\lambda_2) + (1-p_1-p_2)\, f_3(z|\lambda_3)$,
$f(z|\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2) = p_1\lambda_1\exp(-\lambda_1 z) + p_2\frac{z}{\lambda_2^2}\exp\left(-\frac{z^2}{2\lambda_2^2}\right) + (1-p_1-p_2)\,\lambda_3(z+1)^{-(\lambda_3+1)}$,

$$\begin{aligned}\pi_v(\lambda_1,\lambda_2,\lambda_3,p_1,p_2|\mathbf{y}) &= \sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i\end{pmatrix}\begin{pmatrix} i \\ j\end{pmatrix}\exp(-D_{1v}\lambda_1)\exp\left(-D_{2v}\lambda_2^{-2}\right)\\ &\times \frac{\exp(-D_{3v}\lambda_3)\, p_1^{A_{0v}-1} p_2^{B_{0v}-1}(1-p_1-p_2)^{C_{0v}-1}}{\Omega_v \lambda_1^{1-A_{1v}}\lambda_2^{2A_{2v}+1}\lambda_3^{1-A_{3v}}}\end{aligned}$$

So, after simplification, the posterior predictive distribution (28) for a future observation $Z = Y_{n+1}$ given data $\mathbf{y}$ is:

$$\begin{aligned} f(z\,|\mathbf{y}) = {} & \tfrac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i\end{pmatrix}\begin{pmatrix} i \\ j\end{pmatrix}\tfrac{B(A_{0v}+1,\ C_{0v})B(B_{0v},\ A_{0v}+C_{0v}+1)}{(D_{1v}+z)^{(A_{1v}+1)}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}}\\ & + \tfrac{z\,\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{2\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i\end{pmatrix}\begin{pmatrix} i \\ j\end{pmatrix}\tfrac{B(A_{0v},\ C_{0v})B(B_{0v}+1,\ A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}\left(D_{2v}+\frac{z^2}{2}\right)^{(A_{2v}+1)}D_{3v}^{A_{3v}}}\\ & + \tfrac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{2(1+z)\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i\end{pmatrix}\begin{pmatrix} i \\ j\end{pmatrix}\tfrac{B(A_{0v},\ C_{0v}+1)B(B_{0v},\ A_{0v}+C_{0v}+1)}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}\{D_{3v}+\ln(1+z)\}^{(A_{3v}+1)}}. \end{aligned} \tag{29}$$

7.2 The Bayesian Predictive Interval

In order to construct a Bayesian predictive interval, the two endpoints (L and U) of the Bayesian predictive interval can be obtained using the posterior predictive distribution given in (29). A $100(1-\alpha)\%$ Bayesian predictive interval ($L,\ U$) can be obtained by solving the following equations:
$\int_0^L f(z\,|\mathbf{y})\,dz = \frac{\alpha}{2}$ and $\int_U^\infty f(z\,|\mathbf{y})\,dz = \frac{\alpha}{2}$.
After substitution and simplification the above equations, the Bayesian predictive interval ($L,\ U$) can be written as:

$$\begin{aligned} & \tfrac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v A_{1v}}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i\end{pmatrix}\begin{pmatrix} i \\ j\end{pmatrix}\tfrac{B(A_{0v}+1,\ C_{0v})B(B_{0v},\ A_{0v}+C_{0v}+1)}{\left\{D_{1v}^{-A_{1v}}-(D_{1v}+L)^{-A_{1v}}\right\}^{-1}D_{2v}^{A_{2v}}D_{3v}^{A_{3v}}}\\ & + \tfrac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{2\Omega_v A_{2v}}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i\end{pmatrix}\begin{pmatrix} i \\ j\end{pmatrix}\tfrac{B(A_{0v},\ C_{0v})B(B_{0v}+1,\ A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}\left\{D_{2v}^{-A_{2v}}-\left(D_{2v}+\frac{L^2}{2}\right)^{-A_{2v}}\right\}^{-1}D_{3v}^{A_{3v}}}\\ & + \tfrac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{2\Omega_v A_{3v}}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i\end{pmatrix}\begin{pmatrix} i \\ j\end{pmatrix}\tfrac{B(A_{0v},\ C_{0v}+1)B(B_{0v},\ A_{0v}+C_{0v}+1)}{D_{1v}^{A_{1v}}D_{2v}^{A_{2v}}\left\{D_{3v}^{-A_{3v}}-(D_{3v}+\ln(1+L))^{-A_{3v}}\right\}^{-1}} = \frac{\alpha}{2} \end{aligned}$$

and

$$\frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{2\Omega_v A_{1v}}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v}+1,\,C_{0v})B(B_{0v},\,A_{0v}+C_{0v}+1)}{(D_{1v}+U)^{A_{1v}}D_{2v}^{A_{2v}}\,D_{3v}^{A_{3v}}}$$
$$+\frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{2\Omega_v A_{2v}}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},\,C_{0v})B(B_{0v}+1,\,A_{0v}+C_{0v})}{D_{1v}^{A_{1v}}\left(D_{2v}+\frac{U^2}{2}\right)^{A_{2v}}D_{3v}^{A_{3v}}}$$
$$+\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{2\Omega_v A_{3v}}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\begin{pmatrix} n-r \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\frac{B(A_{0v},\,C_{0v}+1)B(B_{0v},\,A_{0v}+C_{0v}+1)}{D_{1v}^{A_{1v}}\,D_{2v}^{A_{2v}}\{D_{3v}+\ln(1+U)\}^{A_{3v}}}=\frac{\alpha}{2}.$$

8 Limiting Expression for Complete Data Set

When test termination time tends to infinity (i.e., $t \to \infty$), uncensored observations r tends to sample size n (i.e., $r \to n$) and r_l tends to unknownn_l (i.e., $r_l \to n_l$), $l = 1,\ 2,\ 3$. Subsequently, all the observations which are slot in our statistical analysis become uncensored and the information contained in the sample is increased. As a result, the posterior risks of the Bayes estimators diminish and efficiency of the Bayes estimators is increased because all the observations are incorporated in sample. The limiting expressions for the Bayes estimators and their variances using the NIP (JP) and the IP are showcased in Tables 1-2.

Table 1: Limiting expressions for the Bayes estimators and their variances as $t \to \infty$ using the JP

Parameters	Bayes Estimators	Variances of Bayes Estimators
λ_1	$\lim\limits_{t\to\infty}\hat{\lambda}_1\vert\mathbf{y}=\frac{n_1}{\sum_{k=1}^{n_1}y_{1k}}$	$\lim\limits_{t\to\infty}Var\left(\hat{\lambda}_1\vert\mathbf{y}\right)=\frac{n_1}{\left(\sum_{k=1}^{n_1}y_{1k}\right)^2}$
λ_2	$\lim\limits_{t\to\infty}\hat{\lambda}_2\vert\mathbf{y}=\frac{\Gamma\left(n_2-\frac{1}{2}\right)}{\left(\frac{1}{2}\sum_{k=1}^{n_2}y_{2k}^2\right)^{-1/2}\Gamma(n_2)}$	$\lim\limits_{t\to\infty}Var\left(\hat{\lambda}_2\vert\mathbf{y}\right)=\frac{\left(\frac{1}{2}\sum_{k=1}^{n_2}y_{2k}^2\right)}{\left[\frac{1}{(n_2-1)}-\left\{\frac{\Gamma\left(n_2-\frac{1}{2}\right)}{\Gamma(n_2)}\right\}^2\right]^{-1}}$
λ_3	$\lim\limits_{t\to\infty}\hat{\lambda}_3\vert\mathbf{y}=\frac{n_3}{\sum_{k=1}^{n_3}\ln(y_{3k}+1)}$	$\lim\limits_{t\to\infty}Var\left(\hat{\lambda}_3\vert\mathbf{y}\right)=\frac{n_3}{\left\{\sum_{k=1}^{n_3}\ln(y_{3k}+1)\right\}^2}$
p_1	$\lim\limits_{t\to\infty}\hat{p}_1\vert\mathbf{y}=\frac{n_1+1}{n+3}$	$\lim\limits_{t\to\infty}Var\left(\hat{p}_1\vert\mathbf{y}\right)=\frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$
p_2	$\lim\limits_{t\to\infty}\hat{p}_2\vert\mathbf{y}=\frac{n_2+1}{n+3}$	$\lim\limits_{t\to\infty}Var\left(\hat{p}_2\vert\mathbf{y}\right)=\frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$

9 Simulation Study

We resort to a simulation study since the Bayes estimators (under different priors and loss functions) cannot be compared analytically. Monte Carlo simulation study is done

Table 2: Limiting expressions for the Bayes estimators and their variances as $t \to \infty$ using the IP

Parameters	Bayes Estimators	Variances of Bayes Estimators
λ_1	$\lim_{t\to\infty} \hat{\lambda}_1 \vert \mathbf{y} = \frac{n_1+a_1}{\sum_{k=1}^{n_1} y_{1k}+b_1}$	$\lim_{t\to\infty} Var\left(\hat{\lambda}_1 \vert \mathbf{y}\right) = \frac{n_1+a_1}{\left(\sum_{k=1}^{n_1} y_{1k}+b_1\right)^2}$
λ_2	$\lim_{t\to\infty} \hat{\lambda}_2 \vert \mathbf{y} = \frac{\Gamma\left(n_2+a_2-\frac{1}{2}\right)}{\left(\frac{1}{2}\sum_{k=1}^{n_2} y_{2k}^2+b_2\right)^{-1/2}\Gamma(n_2+a_2)}$	$\lim_{t\to\infty} Var\left(\hat{\lambda}_2 \vert \mathbf{y}\right) = \frac{\left(\frac{1}{2}\sum_{k=1}^{n_2} y_{2k}^2+b_2\right)}{\left[\frac{1}{(n_2+a_2-1)}-\left\{\frac{\Gamma\left(n_2+a_2-\frac{1}{2}\right)}{\Gamma(n_2+a_2)}\right\}^2\right]^{-1}}$
λ_3	$\lim_{t\to\infty} \hat{\lambda}_3 \vert \mathbf{y} = \frac{n_3+a_1}{\sum_{k=1}^{n_3} \ln(y_{3k}+1)+b_1}$	$\lim_{t\to\infty} Var\left(\hat{\lambda}_3 \vert \mathbf{y}\right) = \frac{n_3+a_3}{\left\{\sum_{k=1}^{n_3} \ln(y_{3k}+1)+b_3\right\}^2}$
p_1	$\lim_{t\to\infty} \hat{p}_1 \vert \mathbf{y} = \frac{n_1+a}{n+a+b+c}$	$\lim_{t\to\infty} Var\left(\hat{p}_1 \vert \mathbf{y}\right) = \frac{(n_1+a)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)}$
p_2	$\lim_{t\to\infty} \hat{p}_2 \vert \mathbf{y} = \frac{n_2+b}{n+a+b+c}$	$\lim_{t\to\infty} Var\left(\hat{p}_2 \vert \mathbf{y}\right) = \frac{(n_2+b)(n_1+n_3+a+c)}{(n+a+b+c)^2(n+a+b+c+1)}$

to investigate the performance of the Bayes estimators and check the impact of various parametric values, different sample sizes and test termination times under different priors. For each of the five parameters λ_1, λ_2, λ_3, p_1 and p_2 of a 3-component mixture of exponential, Rayleigh and Burr Type-XII distributions, we simulated the Bayes estimates and variances through a Monte Carlo simulation using the following steps.

1. A sample of size n from the mixtures may be generated using Mathematica software as follows:

2. Generate $p_1 n$ observations randomly from 1^{st} component density function $f_1(y) = \lambda_1 \exp(-\lambda_1 y)$.

3. Generate $p_2 n$ observations randomly from 2^{nd} component density function $f_2(y) = \frac{y}{\lambda_2^2} \exp\left(-\frac{y^2}{2\lambda_2^2}\right)$.

4. Generate remaining $(1-p_1-p_2)\,n$ observations randomly from 3^{rd} component density function $f_3(y) = \lambda_3 (y+1)^{-(\lambda_3+1)}$.

5. Select a sample censored at a fixed test termination time t. Take observations which are greater than a fixed test termination time t as censored ones.

6. Calculate the Bayes estimate $\hat{\xi}_i$ and variance $Var\left(\hat{\xi}_i\right)$ of a parameter say ξ using the censored sample in solving (7)-(16).

7. Repeat steps 1-3, 1000 times.

8. Calculate the simulated Bayes estimate and its simulated posterior risk as $\hat{\xi} = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\xi}_i\right)$ and $Var\left(\hat{\xi}\right) = \frac{1}{1000}\sum_{i=1}^{1000} Var\left(\hat{\xi}_i\right)$, respectively.

The above steps 1-5 are used for each of the sample sizes n = 30, 50, 100, 200, each choice of the vector of the parameters $(\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2)$ ={(3, 0.4, 5, 0.4, 0.3), (2, 0.3, 4, 0.4, 0.3), (3, 0.4, 5, 0.35, 0.25)} taking test termination times $t = 0.7$ and 1. The choice of the test termination time is made in such a way that the censoring rate in resulting sample remains in between 10% to 25%. The simulated results, so obtained, are arranged in Tables 3-11.

From Tables 3-8, it can be seen that the degree of under-estimation (or/and over-estimation) of the component and proportion parameters (through Bayes estimators) is lesser for larger sample size as compared to smaller sample size at a fixed test termination time. Similarly, the degree of over-estimation (or/and under-estimation) of the component and proportion parameters is greater for smaller test termination time as compared to larger test termination time at a fixed sample size. Also, it is observed that difference of the Bayes estimates from assumed parameters reduce to zero with an increase in sample size for different test termination times. The same observation can be made with larger test termination time as compared to smaller test termination time for varying sample sizes.

Also, it is observed that the variances of Bayes estimators using the NIP and the IP reduce with an increase in sample size at a fixed test termination time. For larger test termination time, the variances of Bayes estimators are lesser than the variances for smaller test termination time irrespective of the prior and sample size. Also, the variances of Bayes estimators of component parameters are smaller (larger) for smaller (larger) component parametric values (with same proportion parametric values) for each sample size and test termination time considered in the simulation study. Similarly, the variances of Bayes estimators of first two component (third component) parameters are larger (smaller) for smaller proportion parametric values (with same component parametric values) for different sample sizes (test termination times) at a fixed test termination time (sample size). Whereas, the variances of Bayes estimators of proportion parameters are smaller for smaller proportion parametric values (with same component parametric values) for varying test termination times (sample sizes) at a fixed sample size (test termination time).

Moreover, it is to be noted that selection of best prior is made based on the variances of Bayes estimators associated with it. The selection of best prior does not depend on sample size and test termination time. As far as the problem of selecting a suitable prior is concerned, it can be seen that the IP occurs as an efficient prior than that using the NIP (JP) due to lesser variances.

The 90% Bayesian predictive intervals using the NIP (JP) and the IP are showcased in Tables 9-11. It is concluded that the Bayesian predictive intervals become narrower with an increase in sample size at a fixed test termination time. The same observation can be made with larger test termination time as compared to smaller test termination time at a fixed sample size. The Bayesian predictive intervals are narrower (wider) for larger (smaller) component parametric values (but proportion parametric values are same) at each sample size and test termination time considered in the simulation study. But, in this study, the Bayesian predictive intervals are wider (narrower) for larger (smaller) proportion parametric values (but component parametric values are same) at different

Table 3: Bayes estimators (BE) and variances (Var) using the JP with $\lambda_1 = 3,\ \lambda_2 = 0.4,\ \lambda_3 = 5,\ p_1 = 0.4,\ p_2 = 0.3$

t	n	BE/Var	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.7	30	BE	3.602320	0.382781	5.920740	0.392154	0.290014
		Var	2.200540	0.012979	6.693950	0.008102	0.007615
	50	BE	3.326910	0.390128	5.591130	0.395441	0.291559
		Var	1.185160	0.007060	3.658000	0.005293	0.005061
	100	BE	3.278390	0.394767	5.292090	0.396697	0.294180
		Var	0.626409	0.003601	1.809800	0.003017	0.002845
	200	BE	3.127070	0.396888	5.090450	0.399077	0.294684
		Var	0.313185	0.001909	0.940550	0.001624	0.001555
1.0	30	BE	3.448250	0.414545	5.794120	0.394755	0.304601
		Var	1.373150	0.007707	5.301490	0.007236	0.006507
	50	BE	3.273830	0.408364	5.321820	0.397129	0.303437
		Var	0.768727	0.004395	2.771770	0.004618	0.004128
	100	BE	3.207630	0.404035	5.143150	0.398862	0.301565
		Var	0.379652	0.002157	1.290720	0.002423	0.002156
	200	BE	3.107890	0.402331	5.104650	0.399522	0.300997
		Var	0.181042	0.001085	0.654712	0.001251	0.001106

sample sizes time. Also, the Bayesian predictive intervals using the IP are narrow than the predictive intervals using the NIP (JP).

10 Estimation under Real Life Data

Consider the mixture data $\mathbf{x}$=($x_{11},\ x_{12},\ ...,\ x_{1r_1},\ x_{21},\ x_{22},\ ...,\ x_{2r_2},\ x_{31},\ x_{32}, ..., x_{3r_3}$), studied by Davis (1952) on lifetimes (in thousand hours) of many components used in aircraft sets. To explain the proposed methodology, we take the data on three different components, namely, R105 Resistor used in PE218 Converter, Z303 Network used in RF Unit and V7 Transmitter Tube. Davis (1952) showed that the mixture data ($\mathbf{x}$) can be modeled by a mixture of exponential distributions. For second component (Z303 Network used in RF Unit), the transformation $y = \sqrt{2x}$ of exponential random data ($\mathbf{x}$) yields the Rayleigh random data ($\mathbf{y}$). For third component (V7 Transmitter Tube), the transformation $y = \exp(x) - 1$ of exponential random data ($\mathbf{x}$) yields the Burr Type-XII random data ($\mathbf{y}$). These transformations support using the real life mixture data for applying the suggested Bayesian analysis. It is unknown that which component fails

Table 4: Bayes estimators (BE) and variances (Var) using the IP with $\lambda_1 = 3,\ \lambda_2 = 0.4,\ \lambda_3 = 5,\ p_1 = 0.4,\ p_2 = 0.3$

t	n	BE/Var	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.7	30	BE	2.356480	0.487003	2.984300	0.386409	0.290600
		Var	0.520248	0.012139	1.095070	0.007944	0.007259
	50	BE	2.567170	0.449516	3.403030	0.388219	0.291903
		Var	0.449817	0.006507	1.036850	0.005133	0.004757
	100	BE	2.774100	0.426028	3.917060	0.392694	0.294621
		Var	0.325301	0.003088	0.854078	0.002873	0.002614
	200	BE	2.920040	0.413346	4.317160	0.396557	0.296583
		Var	0.217843	0.001642	0.609271	0.001546	0.001436
1.0	30	BE	2.413620	0.477529	3.164020	0.386970	0.308796
		Var	0.463332	0.007588	1.070340	0.007149	0.006441
	50	BE	2.624400	0.456875	3.622840	0.391578	0.306765
		Var	0.376156	0.004217	0.960790	0.004561	0.004086
	100	BE	2.833610	0.413381	4.144930	0.393894	0.302609
		Var	0.248801	0.002093	0.717335	0.002410	0.002130
	200	BE	2.932280	0.402285	4.490690	0.397227	0.301344
		Var	0.147921	0.001061	0.468342	0.001247	0.001098

until a failure occurs at or before the test termination time $t = 1$ hour. Thus, we have a type-I right censored data at $t = 1$ hours on $n = 582$ radar sets. To evaluate the Bayes estimates and variances, the summary of real mixture data is:

$$n = 582, n_1 = 317, n_2 = 77, n_3 = 188, r_1 = 252, r_2 = 54, r_3 = 175, r = 481, n - r = 101,$$

$$\sum_{k=1}^{r_1} y_{1k} = 90.60, \sum_{k=1}^{r_2} y_{2k}^2 = 46.40, \sum_{k=1}^{r_3} \ln(1 + y_{3k}) = 46.125.$$

Since $n - r = 101$, we have almost 17.35 percent type-I right censored sample. The Bayes estimates and variances are given in Table 12.

It is observed that results obtained through the real data, given in Table 12, are compatible with simulated results.The performance of the Bayes estimators using the IP is seen as superior than the NIP (JP) based on minimum amount of variances of the Bayes estimators.

The 90% Bayesian predictive intervals $(L,\ U)$ using the JP and the IP are presented in Table 13.

Table 5: Bayes estimators (BE) and variances (Var) using the JP with $\lambda_1 = 2,\ \lambda_2 = 0.3,\ \lambda_3 = 4,\ p_1 = 0.4,\ p_2 = 0.3$

t	n	BE/Var	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.7	30	BE	2.787750	0.334150	4.875380	0.367916	0.319747
		Var	1.719440	0.007329	5.870150	0.008614	0.007592
	50	BE	2.566530	0.320387	4.482780	0.374579	0.314790
		Var	0.941188	0.004175	3.197340	0.005926	0.005054
	100	BE	2.402150	0.313439	4.195270	0.379594	0.311137
		Var	0.435550	0.002049	1.524200	0.003340	0.002671
	200	BE	2.250090	0.310095	4.054360	0.385908	0.306865
		Var	0.200057	0.001048	0.788644	0.001813	0.001368
1.0	30	BE	2.464890	0.318109	4.655310	0.382104	0.307851
		Var	0.937699	0.004793	4.197770	0.007624	0.006499
	50	BE	2.313290	0.313197	4.299610	0.386273	0.305298
		Var	0.502155	0.002528	2.202300	0.004901	0.004047
	100	BE	2.170090	0.306708	4.146010	0.391559	0.302624
		Var	0.217825	0.001062	1.060830	0.002601	0.002066
	200	BE	2.096940	0.303220	4.067390	0.395071	0.301295
		Var	0.100875	0.000476	0.538922	0.001349	0.001044

From Table 13, it can be seen that the 90% Bayesian predictive intervals using the IP are narroweras compared to the Bayesian predictive intervals using the NIP (JP).

11 Conclusions

Monte Carlo simulation study has revealed some important and fascinating properties of the Bayes estimators. The application and importance of mixture models in real life phenomena is un-deniable. A 3-component mixture of exponential, Rayleigh and Burr Type-XII distributions is developed to model lifetime data. Type-I right censoring sampling scheme is considered. Assuming the availability of the NIP and the IP, expressions of the Bayes estimators and their variances are derived. As the cut off test termination time tends to infinity, the limiting expressions (for complete sample) of the Bayes estimators and their variances are greatly simplified. To judge the relative performance of the Bayes estimators and also to deal with the problems of selecting the priors at different sample sizes and test termination times, a comprehensive Monte Carlo simulation has been conducted. From numerical results, it is observed that an increase in sample

Table 6: Bayes estimators (BE) and variances (Var) using the IP with $\lambda_1 = 2,\ \lambda_2 = 0.3,\ \lambda_3 = 4,\ p_1 = 0.4,\ p_2 = 0.3$

t	n	BE/Var	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.7	30	BE	2.254210	0.435999	2.730550	0.350452	0.337713
		Var	0.446858	0.007242	0.969025	0.007973	0.007317
	50	BE	2.174170	0.397820	2.985750	0.356202	0.332550
		Var	0.377069	0.004038	0.870470	0.005312	0.005025
	100	BE	2.142280	0.357479	3.337420	0.364385	0.323096
		Var	0.257236	0.002031	0.699058	0.003019	0.002619
	200	BE	2.115150	0.334042	3.574800	0.375050	0.314347
		Var	0.152578	0.001023	0.500818	0.001702	0.001348
1.0	30	BE	1.875660	0.417382	2.742400	0.372176	0.319168
		Var	0.355903	0.004611	0.887095	0.007341	0.006327
	50	BE	2.098340	0.377392	3.004290	0.375420	0.313072
		Var	0.274356	0.002379	0.773419	0.004805	0.004015
	100	BE	2.081290	0.339673	3.350050	0.383946	0.306685
		Var	0.165103	0.001008	0.571601	0.002566	0.002024
	200	BE	2.063400	0.319568	3.596570	0.390302	0.303144
		Var	0.090325	0.000441	0.384827	0.001347	0.001040

size (test termination time) at a fixed test termination time (sample size) provides improved Bayes estimators. The degree of over-estimation (or/and under-estimation) of the Bayes estimators is quite larger (smaller) for relatively smaller (larger) sample sizes (test termination times) at a fixed test termination time (sample size). Also, as sample size (test termination time) increases (decreases), the variances of Bayes estimators decrease (increase) for a fixed test termination time (sample size). However, the variances of Bayes estimators are smaller (larger) when component (proportion) parameters are relatively smaller (larger) and vice versa. Moreover, the Bayesian predictive intervals become narrower with an increase in sample size (test termination time) at a fixed test termination time (sample size). The Bayesian predictive intervals are narrower (wider) for larger component (proportion) parametric values as compared to smaller component (proportion) parametric values. The Bayesian predictive intervals using the IP are narrow than the predictive intervals using the NIP. The results obtained through real life mixture data coincide with the simulated results. Finally, we conclude that the IP is more efficient and suitable prior for estimating the component and proportion parameters.

Table 7: Bayes estimators (BE) and variances (Var) using the JP with $\lambda_1 = 3,\ \lambda_2 = 0.4,\ \lambda_3 = 5,\ p_1 = 0.35,\ p_2 = 0.25$

t	n	BE/Var	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.7	30	BE	3.611530	0.414533	5.747880	0.326807	0.273771
		Var	2.707290	0.015264	4.722200	0.007419	0.007123
	50	BE	3.460130	0.390867	5.552320	0.336729	0.260259
		Var	1.405980	0.009375	2.623870	0.005117	0.004430
	100	BE	3.257810	0.393032	5.267710	0.348103	0.258355
		Var	0.711027	0.004478	1.333090	0.002765	0.002555
	200	BE	3.140540	0.396598	5.135370	0.349029	0.255785
		Var	0.358139	0.002377	0.686220	0.001509	0.001382
1.0	30	BE	3.532100	0.411617	5.654520	0.331077	0.264611
		Var	1.811260	0.009368	3.585270	0.006712	0.006138
	50	BE	3.279370	0.408845	5.412040	0.346548	0.257306
		Var	0.857538	0.005857	1.998300	0.004432	0.003644
	100	BE	3.189190	0.405310	5.211860	0.348336	0.254082
		Var	0.433161	0.002726	0.955591	0.002306	0.001944
	200	BE	3.117970	0.402108	5.084050	0.349617	0.251756
		Var	0.212387	0.001330	0.467076	0.001180	0.000990

Table 8: Bayes estimators (BE) and variances (Var) using the IP with $\lambda_1 = 3,\ \lambda_2 = 0.4,\ \lambda_3 = 5,\ p_1 = 0.35,\ p_2 = 0.25$

t	n	BE/Var	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.7	30	BE	2.241380	0.500469	3.358090	0.328551	0.271304
		Var	0.534126	0.014324	1.075610	0.007340	0.006807
	50	BE	2.543370	0.469055	3.808280	0.353795	0.243770
		Var	0.473649	0.008778	0.975438	0.004946	0.004215
	100	BE	2.782700	0.426731	4.236040	0.347462	0.248103
		Var	0.365105	0.003840	0.722560	0.002669	0.002348
	200	BE	2.867130	0.410906	4.625040	0.349617	0.247450
		Var	0.239168	0.002084	0.499935	0.001459	0.001299
1.0	30	BE	2.297910	0.489884	3.516950	0.328416	0.280639
		Var	0.486431	0.009151	1.036440	0.006658	0.006104
	50	BE	2.584720	0.466145	3.900530	0.354268	0.251251
		Var	0.400124	0.005604	0.854358	0.004397	0.003636
	100	BE	2.787110	0.429940	4.407530	0.347120	0.254995
		Var	0.272956	0.002632	0.600692	0.002287	0.001921
	200	BE	2.881770	0.414218	4.642020	0.348033	0.252642
		Var	0.162571	0.001299	0.362229	0.001172	0.000984

Table 9: Bayesian predictive intervals using the JP and the IP with $\lambda_1 = 3,\ \lambda_2 = 0.4,\ \lambda_3 = 5,\ p_1 = 0.4,\ p_2 = 0.3$

		JP		IP	
t	n	L	U	L	U
0.7	30	0.018178	1.306350	0.021257	1.190290
	50	0.018405	1.163890	0.020206	1.080560
	100	0.018803	1.070330	0.019618	1.005070
	200	0.019059	1.017940	0.019412	0.960766
1.0	30	0.018116	1.170270	0.021067	1.108610
	50	0.018319	1.070900	0.020118	1.031960
	100	0.018754	1.019800	0.019645	0.980107
	200	0.018948	0.990470	0.019387	0.955550

Table 10: Bayesian predictive interval using the JP and the IP with $\lambda_1 = 2,\ \lambda_2 = 0.3,\ \lambda_3 = 4,\ p_1 = 0.4,\ p_2 = 0.3$

		JP		IP	
t	n	L	U	L	U
0.7	30	0.023290	1.635070	0.026500	1.223850
	50	0.024010	1.430670	0.025794	1.132410
	100	0.024707	1.317630	0.025458	1.066920
	200	0.024990	1.243350	0.025340	1.044110
1.0	30	0.024237	1.563580	0.027261	1.167440
	50	0.024683	1.394570	0.026519	1.086590
	100	0.025155	1.261590	0.026164	1.038770
	200	0.025398	1.230510	0.025945	1.018970

Table 11: Bayesian predictive interval using the JP and the IP with $\lambda_1 = 3,\ \lambda_2 = 0.4,\ \lambda_3 = 5,\ p_1 = 0.35,\ p_2 = 0.25$

		JP		IP	
t	n	L	U	L	U
0.7	30	0.016661	1.287230	0.019179	1.179950
	50	0.016188	1.146600	0.017530	1.074140
	100	0.016792	1.059630	0.017369	0.984689
	200	0.016874	1.007210	0.017114	0.937043
1.0	30	0.016523	1.163370	0.018904	1.103730
	50	0.016308	1.069690	0.017610	1.030570
	100	0.016661	1.012260	0.017317	0.967683
	200	0.016837	0.980143	0.017142	0.926550

Table 12: Bayes estimators (BE) and variances (Var) using the JP and the IP with real life mixture data

Prior	Estimators	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
JP	BE	1.767531	0.769615	3.302097	0.523493	0.163490
	Var	0.040668	0.009974	0.110522	0.000956	0.000779
IP	BE	1.756090	0.773745	3.307278	0.523122	0.164274
	Var	0.036648	0.009053	0.107516	0.000907	0.000729

Table 13: Bayesian predictive interval $(L,\ U)$ using the NIP (JP) and the IP with real life mixture data

JP		IP	
L	U	L	U
0.026608	1.768050	0.027085	1.637290

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