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By Bander Al-Zahrani, Maha Gindwan

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Estimating the parameter of the Lindley distribution under progressive type-II censored data

Bander Al-Zahrani*and Maha Gindwan

King Abdulaziz University
Department of Statistics, Jeddah 21589, Saudi Arabia

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We consider the estimation problem of the Lindley distribution based on progressive type-II censored data. We use the EM algorithm for estimating the involved parameter using the maximum likelihood method. The asymptotic variance of the MLE within the EM framework is obtained. Then, the asymptotic confidence intervals of the parameter are constructed. Finally, a real data set and a simulation study are presented to illustrate the obtained results.

keywords: EM algorithm, Lindley distribution, Maximum likelihood estimators, Progressively type-II censoring.

1 Introduction

In the literature of survival analysis and reliability theory, the exponential distribution is widely used as a model of lifetime data. However, the exponential distribution only provide a reasonable fit for modeling phenomenon with constant failure rates. Distributions like gamma, Weibull and lognormal have become suitable alternatives to the exponential distribution in many practical situations. Ghitany et al. (2008) found that the Lindley distribution can be a better model than one based on the exponential distribution. The Lindley distribution belongs to an exponential family and it can be written as a mixture of an exponential with parameter β and a gamma distribution with parameters $(2, \beta)$.

*Corresponding author:bmalzahrani@kau.edu.sa

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A continuous random variable X is said to have Lindley distribution with a parameter β, we write $X \sim Lin(\beta)$, if its probability density function (pdf) is given by

$$f(x) = \frac{\beta^2}{1+\beta}\,(1+x)\,e^{-\beta x}, \quad x > 0,\ \beta > 0. \tag{1}$$

The pdf given in (1) is a special mixture of Exponential(β) and Gamma(2, β) distributions as

$$f(x) = pf_1(x) + (1-p)f_2(x) \quad x > 0,$$

where $p = \beta/(1+\beta)$, $f_1(x) = \beta\, e^{-\beta x}$ and $f_2(x) = \beta^2 x e^{-\beta x}$. The corresponding cumulative distribution function (cdf) is given by

$$F(x) = 1 - \frac{1+\beta+\beta x}{1+\beta}\, e^{-\beta x}, \quad x > 0,\ \beta > 0. \tag{2}$$

Ghitany et al. (2008) gave a comprehensive study on the properties of the Lindley distribution. Recently, different generalizations and extensions of the Lindley distribution have appeared in the literature. Ghitany et al. (2011) considered a two-parameter weighted Lindley distribution. Zakerzadeh and Mahmoudi (2012) proposed a two-parameter lifetime distribution by compounding Lindley and geometric distributions. Shanker et al. (2013) introduced a two-parameter Lindley distribution of which the one-parameter Lindley distribution is a particular case.

There are several situations in reliability experiments and survival analysis in which some items are terminated from the experiments. Many types of censoring schemes are used in lifetime analysis. Conventional type-I, type-II censoring and the hybrid censoring do not allow for removal of items from the experiment. The type-II progressive censoring scheme, which has this advantage, becomes very popular for the last few years. The progressive type-II censoring scheme can be described as follows: at the time of the first failure, $X_{1:m:n}$, R_1 surviving items are removed randomly from the $n1$ remaining surviving items, at the time of the second failure, $X_{2:m:n}$, R_2 surviving items are removed at random from the $n - R_1 - 2$ remaining items, and so on. At the time of the mth observed failure, the remaining $R_m = n - m - R_1 - \cdots - R_{m-1}$ surviving items are removed from the test. For more details see Balakrishnan and Aggarwala (2000). The parameter estimation for different lifetime distributions have been studied based on progressive type-II censored samples see e.g. Sarhan and Abuammoh (2008). Very recently, Al-Zahrani and Gindwan (2014) considered the problem of parameter estimation of a two-parameter Lindley distribution under hybrid censoring.

The rest of this paper is structured as follows. Section 2 gives the maximum likelihood estimator based on progressively type-II censored sample. Section 3 describes how to obtain the MLE of the unknown parameter of the Lindley distribution using the EM algorithm. Also, the MCMC method is used to generate samples from the Lindley distribution and hence a simulation study is carried out to assess the obtained results. An examples is considered for illustrative purposes. Finally, we conclude in Section 4.

2 Estimation

In this section the maximum likelihood estimator is obtained based on progressively type-II censored Lindley distribution. Suppose n independent items are put on a life test with the corresponding lifetimes $X_1, X_2, \cdots, X_n$ being identically distributed. We assume that X_i, $i = 1, 2, \cdots, n$ are independent identical distributed with pdf (1).

2.1 Maximum likelihood estimation

We consider estimation by the method of maximum likelihood. The likelihood for a random sample $X_1, \cdots, X_n$ from (1) under type-II progressively censored data is given by

$$\begin{aligned} L(X|\,\beta) &= C\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i} \\ &= C\prod_{i=1}^{m} \frac{\beta^2}{\beta+1}\,(1+x_i)\,e^{-\beta x_i}\left[\frac{\beta+1+\beta x_i}{\beta+1}\,e^{-\beta x_i}\right]^{R_i}, \end{aligned}$$

where $C = n(n-R_1-1)(n-R_1-R_2-2)\cdots(n-R_1-\cdots-R_{m-1}-m+1)$. The log likelihood function without the constant term can be written as

$$\begin{aligned} \log L &= 2m\log(\beta) - m\log(\beta+1) + \sum_{i=1}^{m}\log(1+x_i) \\ &\quad + \sum_{i=1}^{m} R_i \log\left(\frac{\beta+1+\beta x_i}{\beta+1}\right) - \beta\sum_{i=1}^{m} x_i(1+R_i). \end{aligned} \tag{3}$$

Differentiating (3) with respect to β yields the likelihood equation for β

$$\frac{\partial \log L}{\partial \beta} = \frac{2m}{\beta} - \frac{m}{\beta+1} - \sum_{i=1}^{m} x_i(1+R_i) + \sum_{i=1}^{m} \frac{R_i\,x_i}{(\beta+1+\beta x_i)(\beta+1)} = 0.$$

Usual algebraic solution for the equation (4) is not working due to the properties of transcendental equation. Therefore, a numerical iteration can be used to solve the above equation. The Newton-type method of maximization can be carried out.

2.2 EM Algorithm

The Expectation-Maximization (EM) algorithm is an applicable technique for parameter estimation by maximum likelihood particularly in incomplete-data problems. The EM algorithm, originally proposed by Dempster et al. (1977). For examples and a good historical account of the EM algorithm see e.g. McLachlan and Krishnan (2007).

Suppose that $\mathbf{X}$ is a random vector with a joint density $f(x;\theta)$ and a p-dimensional parameter $\theta \in \Theta$. If $\mathbf{X}$ were observed, then the maximum likelihood estimators of θ based on the distribution of $\mathbf{X}$ can be obtained by maximizing the log-likelihood function of $\mathbf{X}$.

However, if only some of the complete-data vector $\mathbf{X}$ is observed. We will denote this by expressing $\mathbf{X}$ as $(\mathbf{Y};\mathbf{Z})$, where $\mathbf{Y}$ denotes the observed and $\mathbf{Z}$ denotes the unobserved or missing data. For simplicity, we assume that the missing data are missing at random, so that

$$f(y,z;\theta) = f_1(y;\theta)\ f_2(z|y;\theta),$$

where f_1 is the joint density of Y and f_2 is the joint density of Z given the observed data Y, respectively. Thus, the observed data log-likelihood.

$$l_c(\theta;y) = l(\theta;x) - \log f_2(z|y;\theta),$$

where $l(\theta;x)$ is the log-likelihood function of X. EM algorithm has become useful tool when maximizing l_c is difficult but maximizing the complete-data log-likelihood l is simple. However, since X is not observed, l cannot be evaluated and hence maximized. The EM algorithm attempts to maximize $l(\theta;x)$ iteratively, by replacing it by its conditional expectation given the observed data y. This expectation is computed with respect to the distribution of the complete-data evaluated at the current estimate of θ. More specifically, if $\theta^{(0)}$ is an initial value for θ, then on the first iteration it is required to compute

$$Q(\theta,\theta^{(0)}) = E_{\theta^{(0)}}[l(\theta;x)|y].$$

$Q(\theta,\theta^{(0)})$ is now maximized with respect to θ, that is, $\theta^{(1)}$ is found such that

$$Q(\theta^{(1)},\theta^{(0)}) \geq Q(\theta,\theta^{(0)}).$$

Thus the EM algorithm consists of an E-step (Estimation step) followed by an M-step (Maximization step) defined as follows:

E-step: Compute $Q(\theta;\theta^{(t)})$ where $Q(\theta;\theta^{(t)}) = E_{\theta^{(t)}}[l(\theta;x)|y]$.

M-step: Find $\theta^{(t+1)}$ such that $Q(\theta^{(t+1)};\theta^{(t)}) \geq Q(\theta;\theta^{(t)})$ for all θ.

Let us denote the observed and the censored data by $Y = (Y_1, Y_2 \cdots, Y_m)$ and $Z = (Z_1, \cdots, Z_{n-m})$ respectively. The censored data vector Z can be thought of as missing data. The combination of Y and Z, say $W = (Y,Z)$, forms the complete data set. If we denote the log-likelihood function of the complete data set by $l_c(W;\beta)$, ignoring the additive constant, then we have

$$\begin{aligned} l_c(W;\beta) \quad = \quad & 2n\log(\beta) - n\log(\beta+1) + \sum_{i=1}^{m}\log(1+y_i) - \beta\sum_{i=1}^{m} y_i(1+R_i) \\ & + \sum_{i=1}^{m} R_i \log\left(\frac{\beta+1+\beta y_i}{\beta+1}\right) + \sum_{i=1}^{n-m}\log(1+z_i) \\ & -\beta\sum_{i=1}^{n-m} z_i(1+R_i) + \sum_{i=1}^{n-m} R_i\log\left(\frac{\beta+1+\beta z_i}{\beta+1}\right). \end{aligned} \tag{4}$$

For the E-step of the EM algorithm, we need to compute the log-likelihood function, say $l_s(W;\beta)$ where $l_s(\beta) = E(l_c(W;\beta \mid Y = y)$. Therefore,

$$\begin{aligned} l_s(W;\beta) &= 2n\log(\beta) - n\log(\beta+1) + \sum_{i=1}^{m}\log(1+y_i) \\ &\quad -\beta\sum_{i=1}^{m} y_i(1+R_i) + \sum_{i=1}^{m} R_i\log\left(\frac{\beta+1+\beta y_i}{\beta+1}\right) \\ &\quad +(n-m)\left[A(y_m;\beta) + B(y_m;\beta) - \beta C(y_m;\beta)\right], \end{aligned} \tag{5}$$

where

$$\left.\begin{aligned} A(y_m;\beta) &= E[\ln(1+Z_i)|Z_i > y_m] \\ B(y_m;\beta) &= E[R_i\ln(\tfrac{\beta+1+\beta Z_i}{\beta+1})|Z_i > y_m] \\ C(y_m;\beta) &= E[Z_i(1+R_i)|Z_i > y_m] \end{aligned}\right) \tag{6}$$

To find these terms we first give the following theorem: Given a random variable X distributed according to Lindley distribution, $X \sim Lin(\beta)$, the conditional distribution of Z_i for $i = 1, 2, \cdots, n-m$ is

$$f_{Z|Y}(Z_i|Y_{(1)} = y_{(1)}, \cdots, Y_{(i)} = y_{(i)}) = \frac{\beta^2(1+z_i)\, e^{-\beta z_i}}{(\beta+1+\beta y_m)e^{-\beta y_m}},\ z_i > y_m. \tag{7}$$

Here, Z_i and Z_j for $i \neq j$ are conditionally independent. The proof can be obtained similarly as in Ng et al. (2002). Based on (6) and (7), we can write

$$\begin{aligned} A(y_m;\beta) &= \frac{\beta^2}{D}\int_{y_m}^{\infty}\ln(1+z)\ (1+z)\ e^{-\beta z}dz, \\ B(y_m;\beta) &= \frac{\beta^2}{D}\int_{y_m}^{\infty} R\ln\left(\frac{\beta+1+\beta z}{\beta+1}\right)(1+z)\ e^{-\beta z}dz, \\ C(y_m;\beta) &= \frac{\beta^2}{D}\int_{y_m}^{\infty} z\,(1+R)(1+z)\ e^{-\beta z}dz, \end{aligned}$$

where $D = (\beta+1+\beta y_m)e^{-\beta y_m}$. Following Geddes et al. (1990), we have: $\int_u^\infty t^{s-1}\ln(t)\ e^{-t}dt = [(d/da)\Gamma(a,u)]_{a=s}$. After some transformation we get

$$\begin{aligned} A(y_m;\beta) &= \frac{e^{\beta}}{D}\Bigg\{ -\ln(\beta)\ \Gamma\left(2, \beta(1+y_m)\right) + \left[\frac{d}{da}\Gamma\left(a, \beta(1+y_m)\right)\right]_{a=2}\Bigg\}, \\ B(y_m;\beta) &= \frac{R}{D}\Bigg\{e^{\beta+1}\left(\frac{d}{da}\Gamma\Big[a, (\beta+1+\beta y_m)\Big]_{a=2} - \int_{\beta+1+\beta y_m}^{\infty}\ln(u)e^{-u}du\right) \\ &\quad -\ln(\beta+1)\Big[\beta\, e^{-\beta y_m} + \Gamma(2,\beta y_m)\Big]\Bigg\}, \\ C(y_m;\beta) &= \frac{(1+R)}{D}\left\{\Gamma(2,\beta y_m) + \frac{1}{\beta}\Gamma(3,\beta y_m)\right\}. \end{aligned}$$

Now the M-step involves the maximization of (5) with respect to β. If at the k-th stage the estimate of β is $\hat{\beta}_k$, then $\hat{\beta}_{k+1}$ can be obtained by maximizing

$$\begin{aligned} g(\beta) &= 2n\ln(\beta) - n\ln(\beta+1) + \sum_{i=1}^{m}\log(1+y_i) \\ &\quad + \sum_{i=1}^{m} R_i \log\left(\frac{\beta+1+\beta y_i}{\beta+1}\right) - \beta\sum_{i=1}^{m} y_i(1+R_i) \\ &\quad +(n-m)\left[A(y_m;\hat{\beta}_k) + B(y_m;\hat{\beta}_k) - \beta C(y_m;\hat{\beta}_k)\right]. \end{aligned} \tag{8}$$

The first-order derivative of (8) with respect to β is

$$\begin{aligned} \frac{\partial g}{\partial \beta} &= \frac{2n}{\beta} - \frac{n}{\beta+1} + \sum_{i=1}^{m}\frac{R_i(1+y_i)}{\beta+1+\beta y_i} - \sum_{i=1}^{m}\frac{R_i}{\beta+1} \\ &\quad - \sum_{i=1}^{m} y_i(1+R_i) - (n-m)\ C(y_m;\hat{\beta}_k) = 0. \end{aligned}$$

First find $\hat{\beta}_{k+1}$ by solving a fixed point type equation as $h(\beta) = \beta$. The function $h(\beta)$ is defined as follows:

$$h(\beta) = \frac{n(\beta+2)}{\beta+1}\left[-\sum_{i=1}^{m}\frac{R_i(1+y_i)}{\beta+1+\beta y_i} + \sum_{i=1}^{m}\frac{R_i}{\beta+1} + \sum_{i=1}^{m} y_i(1+R_i) + (n-m)\ C(y_m;\hat{\beta}_{(k)})\right]^{-1}.$$

The E-step and the M-step are then repeated till convergence is achieved to the desired level of accuracy.

2.3 Asymptotic variance

Now, we compute the variance of the MLE of the involved parameter under progressive type-II censoring assuming the EM algorithm is used. We refer to Louis (1982) who developed a procedure for extracting the observed information matrix in incomplete data problem. The idea is as follows.

$$I_Y(\beta) = I_W(\beta) - I_{Z|Y}(\beta),$$

where $I_Y(\beta)$ is the observed information, $I_W(\beta)$ is the complete information and $I_{Z|Y}(\beta)$ is the missing information. Complete information and the missing information are given respectively as:

$$I_W(\beta) = -E\left[\frac{\partial^2 l_c(W,\beta)}{\partial\beta^2}\right]. \tag{9}$$

$$I_{Z|Y}(\beta) = -(n-m)E\left[\frac{\partial^2 \ln f_Z(Z|Y,\beta)}{\partial\beta^2}\right]. \tag{10}$$

For a Lindley distribution the information about β based on the complete data is given by

$$\begin{aligned} I_W(\beta) &= \frac{2n}{\beta^2} - \frac{n}{(\beta+1)^2} + \sum_{i=1}^{m} R_i E\left[\frac{(1+y_i)^2}{(\beta+1+\beta y_i)^2}\right] - \sum_{i=1}^{m} \frac{R_i}{(\beta+1)^2} \\ &+ \sum_{i=1}^{n-m} R_i E\left[\frac{(1+z_i)^2}{(\beta+1+\beta z_i)^2}\right] - \sum_{i=1}^{n-m} \frac{R_i}{(\beta+1)^2}. \end{aligned}$$

Using (7), the logarithm of the conditional distribution is

$$\ln f_{Z|Y}(z) = 2\ln(\beta) + \ln(1+z) - \beta z - \ln(\beta+1+\beta y_m) + \beta y_m. \tag{11}$$

The second order derivative of (11) with respect to β is

$$\frac{\partial^2 \ln f_Z}{\partial \beta^2} = \left(\frac{1+y_m}{\beta+1+\beta y_m}\right)^2 - \frac{2}{\beta^2}.$$

Now we can present $I_{Z|Y}$ using (10).

$$I(Z|Y)(\beta) = (n-m)\left\{\frac{2}{\beta^2} - \left(\frac{1+y_m}{\beta+1+\beta y_m}\right)^2\right\}. \tag{12}$$

The asymptotic variance of $\hat{\beta}$ can be obtained by inverting $I_Y(\beta)$. The asymptotic confidence interval can be readily obtained by using the asymptotic variance of the MLE through the observed information.

3 Simulations and Data Analysis

3.1 Simulation Study

A Markov chain Monte Carlo (MCMC) algorithm is followed to get samples from Lindley distribution. Instead of drawing direct samples from Lindley distribution, we may use the Metropolis-Hastings algorithm. This method is an extension of the usual rejection-acceptance sampling method. For a comprehensive treatment on MCMC methods, Metropolis-Hastings algorithm, one may refer to the book by Robert and Casella (2013). The algorithm is proposed as follows:

Step 1. Specify the values of n and β.

Step 2. Select starting point $x^{(t)}$ satisfying that $f(x^{(t)}) > 0$.

Step 3. Generate Y_t from a proposal distribution $q(y|\, x^{(t)})$, where q is symmetric.

Step 4. Take $X^{(t+1)} = Y_t$ with probability $\rho(x^{(t)}, Y_t)$ or otherwise take $X^{(t+1)} = x^{(t)}$ with probability $1 - \rho(x^{(t)}, Y_t)$, where f is the target density, and

$$\rho(x, y) = \min\left\{\frac{f(y)\, q(x|y)}{f(x)\, q(y|x)}, 1\right\}.$$

Step 5. Compute the MLE using `nlm()` function of `R` package.

Step 6. Compute the MLE using the EM algorithm given in subsection 2.2.

Now we discuss the results of a simulation study based on 5000 simulations to evaluate the MLE. The simulations were performed by using the R software. The sample sizes used in the simulation study are 10(10)100 and the values of β are 1.0 and 1.5 for different censoring schemes. The simulated values of MLE, mean squared error (MSE) and asymptotic confidence intervals for β using Newton-type method and EM algorithm are presented in tables 1 and 2. It can be observed that in most cases the estimators obtained by the EM method converge to the true values of β better than that obtained by the Newton-type method. For the complete case, it can be noted that both methods converge to a very close estimate. A 95% confidence intervals for the parameter are constructed based on asymptotic variance. It is observed that the confidence intervals obtianed by Newton-type are wider than those obtained by EM algorithm. In general, we can not say that EM algorithm is always better than Newton-type method. This depends on the expectation of the log-likelihood function. A combination of the two methods, say Newton-EM, would give better results.

3.2 Data Analysis

For illustrative purpose, we use a data set corresponding to waiting times (in minutes) before service of 100 bank customers as discussed by Ghitany et al. (2008). The data are given as follows: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

For analyzing this data set with progressively type-II censoring, we formulate different censoring schemes for different values of m as shown in Table 3. Empirically, it can be observed from Table 3 that the MLEs obtained by the EM algorithm have the smallest standard errors (SE) as compared with that obtained by using Newton-type method. For fixed m, one can determine the censoring scheme that is most efficient from others.

4 Conclusion

In this study some results on statistical inference were developed using EM algorithm. Under the progressively type-II censoring scheme, we obtain the MLE for the unknown parameter for the Lindley distribution. The MCMC methods are used to generate sample from Lindley distribution. The asymptotic variance of the MLE within the EM framework has been obtained. Consequently, the asymptotic confidence intervals of the parameter have been constructed. Comparisons are made between Newton-type method and EM algorithm via a simulation study. It has been observed in this study that

			Newton-type Method			EM Method		
n	m	R	MLE	MSE	(LCI,UCI)	MLE	MSE	(LCI,UCI)
10	5	(1,0,0,1,3)	0.8897	5.9346	(0.3105, 1.4690)	0.5897	0.2879	(0.2011, 1.0082)
	5	(0,0,0,5,0)	0.8937	2.6107	(0.3144, 1.4729)	0.5952	0.2845	(0.2031, 1.0072)
	7	(0,2,0*3,1,0)	0.9207	1.7175	(0.4073, 1.4342)	0.6515	0.2477	(0.2938, 1.0091)
	7	(0*3,3,0*3)	0.9352	3.9959	(0.4117, 1.4586)	0.6506	0.2494	(0.2933, 1.0079)
20	5	(5,0,5,0,5)	1.4921	52.322	(0.4509, 2.5334)	0.7276	0.2253	(0.1907, 1.2645)
	10	(2,3,0*7,5)	1.1329	0.9683	(0.6090, 1.6568)	0.7163	0.1936	(0.3807, 1.0519)
	15	(3,0*13,2)	1.0423	0.3755	(0.6457, 1.4390)	0.7605	0.1690	(0.4739, 1.0471)
	17	(3,0*16)	1.0794	0.3759	(0.6852, 1.4736)	0.8443	0.1595	(0.5435, 1.1450)
30	10	(4,6,0*7,10)	1.2873	4.4132	(0.6881, 1.8865)	0.7599	0.1554	(0.3867, 1.1332)
	15	(5,0*6,5,0*6,5)	1.2296	0.7538	(0.7640, 1.6952)	0.7630	0.1490	(0.4710, 1.0550)
	20	(5,0*10,5,0*8)	1.2872	0.6960	(0.8538, 1.7206)	0.8142	0.1366	(0.5476, 1.0807)
	25	(1,2,0*22,3)	1.0236	0.2183	(0.7233, 1.3238)	0.8063	0.1369	(0.5704, 1.0422)
40	15	(53,0*11,10)	1.3995	1.4857	(0.8654, 1.9336)	0.8011	0.1201	(0.4853, 1.1169)
	25	(5,0,0,5,0*20,5)	1.2747	0.4843	(0.8964, 1.6530)	0.8135	0.1148	(0.5751, 1.0520)
	35	(2,0*10,1,0*22,2)	1.0794	0.1966	(0.8088, 1.3500)	0.8761	0.1104	(0.6585, 1.0937)
50	25	(10,0,0,5,0*20,10)	1.3482	0.5441	(0.9513, 1.7451)	0.8162	0.0985	(0.5743, 1.0582)
	35	(3,0,7,0*29,1,1,3)	1.2600	0.3632	(0.9431, 1.5770)	0.8432	0.0990	(0.6346, 1.0518)
	45	(3,2,0*43)	1.1645	0.2281	(0.9025, 1.4264)	0.9605	0.1030	(0.7488, 1.1722)
60	30	(5,10,0*26,7,8)	1.2898	0.3922	(0.9470, 1.6325)	0.8099	0.0913	(0.5908, 1.0290)
	40	(2,3,0*10,10,0*26,5)	1.3000	0.3612	(0.9947, 1.6053)	0.8491	0.0881	(0.6525, 1.0457)
	50	(0*48,5,5)	1.0027	0.1143	(0.7982, 1.2073)	0.7952	0.1026	(0.6312, 1.0012)
70	35	(3,7,0*15,5,0*15,10*2)	1.2125	0.2718	(0.9184, 1.5066)	0.7975	0.0898	(0.5978, 1.0573)
	45	(5,0*39,5,0,0,5,10)	1.0987	0.1623	(0.8645, 1.3329)	0.7764	0.1008	(0.6073, 1.0255)
	60	(0*57,2,4,4)	1.0041	0.1000	(0.8165, 1.1916)	0.8102	0.0924	(0.6575, 1.0029)
80	45	(10,0*41,5,10,10)	1.1622	0.1897	(0.9145, 1.4098)	0.7904	0.0889	(0.6172, 1.0036)
	65	(0*30,5,10,0*33)	1.1943	0.2002	(0.9739, 1.4147)	0.8888	0.0755	(0.7272, 1.0504)
	75	(0*74,5)	1.0122	0.0882	(0.8407, 1.1837)	0.8929	0.0747	(0.7412, 1.0447)
90	40	(56,0*32,10,10)	1.4346	0.4668	(0.9517, 1.7675)	0.8495	0.0626	(0.6485, 1.0506)
	60	(5,5,10,10,0*56)	1.6329	0.7324	(0.9968, 1.9589)	0.9365	0.0554	(0.7590, 1.1139)
	80	(0*78,5,5)	1.0158	0.0850	(0.8507, 1.1809)	0.8428	0.0774	(0.7049, 1.0007)
100	50	(10,7,0*46,8,25)	1.2053	0.1865	(0.9621, 1.4485)	0.8054	0.0728	(0.6367, 1.0041)
	70	(10,7,0*66,8,5)	1.2625	0.2215	(0.9399, 1.4852)	0.8555	0.0664	(0.7060, 1.0050)
	90	(1,3,0*50,1,5,036)	1.1452	0.1358	(0.9648, 1.3256)	0.9462	0.0664	(0.7992, 1.0932)
	100	(0*100)	1.0210	0.0764	(0.8689, 1.1732)	1.0210	0.0764	(0.8689, 1.1731)

Table 1: Average estimators, MSE and average asymptotic confidence intervals for $\beta = 1$ at different values of n and m under Newton-type method and EM algorithm.

			Newton Method			EM Method		
n	m	R	MLE	MSE	(LCI,UCI)	MLE	MSE	(LCI,UCI)
10	5	(1,0,0,1,3)	1.0878	11.676	(0.3559, 1.8196)	0.6507	0.8643	(0.2190, 1.5823)
	5	(0,0,0,5,0)	1.1650	20.318	(0.3660, 1.9640)	0.6474	0.8714	(0.2182, 1.5766)
	7	(0,2,0*3,1,0)	1.0284	1.1927	(0.4513, 1.6056)	0.7169	0.7621	(0.3211, 1.6127)
	7	(0*3,3,0*3)	1.0482	1.4197	(0.4579, 1.6385)	0.7208	0.7638	(0.3224, 1.6193)
20	5	(5,0,5,0,5)	1.9666	45.074	(0.5546, 3.3786)	0.8202	0.6313	(0.2035, 1.9369)
	10	(2,3,0*7,5)	1.5087	4.6177	(0.7814, 2.2361)	0.8308	0.5833	(0.4370, 1.7246)
	15	(3,0*13,2)	1.3444	1.2181	(0.8179, 1.8710)	0.9049	0.5195	(0.5597, 1.7502)
	17	(3,0*16)	1.3732	0.7817	(0.8606, 1.8859)	1.0141	0.4394	(0.6480, 1.8801)
30	10	(4,6,0*7,10)	1.9550	18.672	(0.9862, 2.9239)	0.8924	0.4785	(0.4455, 1.8394)
	15	(5,0*6,5,0*6,5)	1.7509	3.2343	(1.0565, 2.4453)	0.9229	0.4442	(0.5640, 1.7817)
	20	(5,0*10,5,0*8)	1.7528	1.3601	(1.1435, 2.3621)	0.9970	0.3859	(0.6656, 1.8284)
	25	(1,2,0*22,3)	1.3191	0.4393	(0.9231, 1.7151)	0.9927	0.4032	(0.6981, 1.7873)
40	15	(53,0*11,10)	2.0933	18.183	(1.2476, 2.9389)	0.9501	0.3881	(0.5681, 1.8321)
	25	(5,0,0,5,0*20,5)	1.7877	1.0003	(1.2373, 2.3381)	1.0137	0.3396	(0.7114, 1.8159)
	35	(2,0*10,1,0*22,2)	1.4354	0.3537	(1.0666, 1.8043)	1.1121	0.2926	(0.8312, 1.8931)
50	25	(10,0,0,5,0*20,10)	1.9763	1.5826	(1.3663, 2.5863)	1.0084	0.3165	(0.7035, 1.8132)
	35	(3,0,7,0*29,1,1,3)	1.7838	0.7268	(1.3190, 2.2485)	1.0764	0.2762	(0.8053, 1.8475)
	45	(3,2,0*43)	1.5841	0.3916	(1.2183, 1.9499)	1.2394	0.2210	(0.9610, 1.9878)
60	30	(5,10,0*26,7,8)	1.8754	1.0685	(1.3529, 2.3978)	1.0049	0.3080	(0.7276, 1.7822)
	40	(2,3,0*10,10,0*26,5)	1.8563	0.7436	(1.4034, 2.3092)	1.0804	0.2600	(0.8255, 1.8354)
	50	(0*48,5,5)	1.3699	0.2385	(1.0821, 1.6577)	1.0444	0.3028	(0.8249, 1.7639)
70	35	(3,7,0*15,5,0*15,10*2)	1.7636	0.6300	(1.3153, 2.2120)	1.0026	0.3005	(0.7463, 1.7589)
	45	(5,0*39,5,0,0,5,10)	1.5649	0.3261	(1.2176, 1.9122)	1.0103	0.3047	(0.7859, 1.7346)
	60	(0*57,2,4,4)	1.3949	0.1976	(1.1264, 1.6634)	1.0861	0.2592	(0.8773, 1.7950)
80	45	(10,0*41,5,10,10)	1.6734	0.3930	(1.3006, 2.0462)	1.0122	0.2893	(0.7859, 1.7386)
	65	(0*30,5,10,0*33)	1.7271	0.3852	(1.3968, 2.0575)	1.1880	0.1874	(0.9674, 1.9086)
	75	(0*74,5)	1.3883	0.1654	(1.1469, 1.6298)	1.1983	0.1897	(0.9904, 1.9063)
90	40	(56,0*32,10,10)	2.1402	1.2169	(1.2176, 2.6628)	1.0418	0.2507	(0.7896, 1.7939)
	60	(5,5,10,10,0*56)	2.4685	1.7130	(1.4558, 2.9811)	1.2100	0.1455	(0.9756, 1.9444)
	80	(0*78,5,5)	1.4099	0.1610	(1.1740, 1.6458)	1.1354	0.2144	(0.9457, 1.8251)
100	50	(10,7,0*46,8,25)	1.7615	0.4107	(1.3885, 2.1345)	1.0181	0.2698	(0.8001, 1.8360)
	70	(10,7,0*66,8,5)	1.8487	0.4244	(1.4095, 2.1879)	1.1326	0.1903	(0.9304, 1.8348)
	90	(1,3,0*50,1,5,0*36)	1.6427	0.2362	(1.3756, 1.9097)	1.2909	0.1391	(1.0856, 1.9962)
	100	(0*100)	1.4158	0.1348	(1.1994, 1.6322)	1.4158	0.1348	(1.1994, 1.9992)

Table 2: Average estimators, MSE and average asymptotic confidence intervals for $\beta = 1.5$ at different values of n and m under Newton method and EM algorithm.

		Newton-type Method			EM Method		
m	R	MLE	SE	(LCI,UCI)	MLE	SE	(LCI,UCI)
50	(10,0*47,20,20)	0.2574	0.0236	(0.2112, 0.3036)	0.1862	0.0187	(0.1495, 0.2229)
70	(5,0*65,2,3,10,10)	0.2399	0.0193	(0.2020, 0.2777)	0.1717	0.0135	(0.1453, 0.1981)
90	(0*89,10)	0.2084	0.0153	(0.1785, 0.2383)	0.1665	0.0124	(0.1422, 0.1908)
100	(0*100)	0.1866	0.0133	(0.1605, 0.2126)	0.1866	0.0133	(0.1606, 0.2126)

Table 3: MLE corresponding SE and 95% asymptotic confidence interval for β at different values of m (observed data) under Newton-type method and EM algorithm.

the numerical MLEs obtained by the EM method converge to the true values of the unknown parameter better than those obtained by the Newton-type method. Also, it has been observed that the coverage probabilities for both methods are better when we had a larger number of observed data. Finally, a practical example has been analyzed for illustrative purpose. We have observed that the EM-algorithm and the Newtontype method produced very satisfactory results, but EM method provided better estimates. Therefore, we can conclude that EM algorithm seems a good procedure in estimation problem.

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