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Analysis of doubly censored Burr type II distribution: a Bayesian look

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Burr model is especially suitable for the life-testing of the products that age with time. Trimmed samples are widely utilized in several areas of statistical practice, especially when some sample values at either or both extremes might have been adulterated. In this article, the problem of estimating the parameter of Burr distribution type II based on trimmed samples under informative and uninformative has been addressed. The problem discussed using Bayesian approach to estimate the shape parameter of Burr type II distribution. Elicitation of hyperparameter through prior predictive approach has also been discussed. Posterior predictive distributions along with posterior predictive intervals and credible intervals have also been derived under different priors. A comparison has been made using the Monte Carlo simulation. A real life data example has also been discussed.

Keywords: Inverse Transformation Method, Doubly Censored Samples, Loss Functions, Posterior Predictive distributions, Credible Intervals.

1 Introduction

Burr [9] introduced a family of twelve cumulative distribution functions for modeling lifetime data. The two important members of the family are Burr types II and XII. The two important distributions, Burr type II and Burr type XII, are interrelated through simple transformation. Trimmed samples are widely employed in several areas of statistical practice, especially when some sample values at either or both extremes might have

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been contaminated. The problem of estimating the parameters of Burr distribution type II based on a trimmed sample and prior information has been considered in this paper. Many authors have discussed different methods of estimation for Burr type XII distribution. Al-Hussaini and Jaheen [1] and Al-Hussaini et al. [3] used different methods for obtaining. Bayes estimates of the shape parameters, reliability and failure rate functions based on type II censored samples. Al-Hussaini et al. [2] studied the maximum likelihood, uniformly minimum variance unbiased, Bayes and empirical Bayes estimators for the parameter k and reliability function when c is known. Wingo [22] derived the theory for the ML point estimation of the parameters of the Burr distribution when Type II singly censored sample is at hand. Ali-Mousa [4] obtained empirical Bayes estimation of the parameter and the reliability function based on accelerated Type II censored data. Gupta et al. [12] discusses analysis of failure time data by Burr distribution. Wang et al. [21] derived the maximum likelihood estimation of the Burr XII distributions parameter with censored and uncensored data. Ali-Mousa and Jaheen [5] explore the maximum likelihood and Bayes estimates for two parameters and the reliability function of the Burr Type XII distribution based on progressive type II censored samples. Feroze and Aslam [11] studied Bayesian analysis of Gumbel type II distribution under doubly censored samples using different loss functions. The random variable X has Burr type II distribution and its pdf is given by

$$f(x,\theta) = \theta e^{-x}\left(1+e^{-x}\right)^{-(\theta+1)}, \qquad -\infty < x < \infty, \quad \theta > 0, \tag{1}$$

then the distribution function of the corresponding distribution is

$$F(x,\theta) = \left(1+e^{-x}\right)^{-\theta}, \qquad \theta > 0. \tag{2}$$

The objective of this paper is to obtain the estimators of the unknown shape parameters of Burr type II based on doubly censored type II. The rest of paper is organized as follows. In section 2, the posterior distributions have been derived under non-informative and informative priors. Estimation of shape parameter has been discussed in section 3. Method of Elicitation of the hyper-parameters via prior predictive approach has been discussed in section 4. Posterior predictive distribution and posterior predictive intervals and credible intervals have been derived in section 5. Simulation study has been performed in section 6. The conclusions regarding the study have been presented in section 7.

2 Prior and Posterior Distributions

Doubly type II censoring is used when the samples are censored at two test termination points, that is, the observations below and above a particular point cannot either be observed or not feasible to be observed. The likelihood function under the doubly type II censored samples can be defined as:

Consider a random sample of size 'n' from Burr type II distribution, and let $x_{(r)} \leq x_{(r+1)} \leq ... \leq x_{(n-s)}$ be the ordered observations that can only be observed. The remaining the "$w-1$" smallest observations and the "$n-s$" largest observations have been censored. Then the likelihood function for the Type II doubly censored sample $x_{(r)} \leq x_{(r+1)} \leq ... \leq x_{(n-s)}$ takes the following form:

$$L(x,\theta) = \frac{n!}{(r-1)!\,(n-s)!}\Pi_{i=r}^{s} f(x_{(i)},\theta)\left\{F(x_{(r)},\theta)\right\}^{r-1}\left\{1-F(x_{(s)},\theta)\right\}^{n-s},$$

$$L(x,\theta) = \frac{n!}{(r-1)!\,(n-s)!}\Pi_{i=r}^{s}\left\{\theta e^{-x_{(i)}}\left(1+e^{-x_{(i)}}\right)^{-(\theta+1)}\right\}$$
$$\left\{\left(1+e^{-x_{(r)}}\right)^{-\theta}\right\}^{r-1}\left\{1-\left(1+e^{-x_{(s)}}\right)^{-\theta}\right\}^{n-s},$$

$$L(x,\theta) \propto \sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\theta^{s-r+1}e^{-\theta\left\{\sum_{i=r}^{s}\ln\left(1+e^{-x_{(i)}}\right)+(r-1)\ln\left(1+e^{-x_{(r)}}\right)+k\ln\left(1+e^{-x_{(s)}}\right)\right\}},$$

$$L(x,\theta) \propto \sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\theta^{R}e^{-\theta\psi_{i,r,k}}, \quad (3)$$

where $\psi_{i,r,k} = \left\{\sum_{i=r}^{s}\ln\left(1+e^{-x_{(i)}}\right)+(r-1)\ln\left(1+e^{-x_{(r)}}\right)+k\ln\left(1+e^{-x_{(s)}}\right)\right\}$ and $R = s-r+1$.

2.1 Posterior Distributions Under Uninformative Prior

The uniform distribution is assumed to be:

$$p(\theta) \propto 1, \qquad \theta > 0. \quad (4)$$

The Jeffreys prior is derived to be:

$$p(\theta) \propto \frac{1}{\theta}, \qquad \theta > 0. \quad (5)$$

The informative prior for the parameter θ is assumed to be exponential distribution:

$$p(\theta) = we^{-\theta w}, \qquad \theta > 0. \quad (6)$$

The informative prior for the parameter θ is assumed to be gamma distribution:

$$p(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, \qquad \theta > 0. \quad (7)$$

Now, the generalized expressions for posterior distributions, Bayesian estimators, posterior risks, posterior predictive distributions, posterior predictive intervals and credible intervals have been presented in the following. The expressions under priors given in (4), (5), (6) and (7) can be derived by putting $j=0, l=0, m=0$, $j=1, l=0, m=0$, $j=0, l=1, m=0$, $j=0, l=0, m=1$, respectively in the generalized results.

Combining the prior distribution and the likelihood function (3), the generalized posterior density of θ is derived as:

$$p(\theta|\,x) = \frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\theta^{R-j+m(a-1)}e^{-\theta\left(\psi_{i,r,k}+lw+mb\right)}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}, \quad (8)$$

3 Estimation of Parameter

From a decision-theoretic view point, in order to select the best estimator, a loss function must be specified and is used to represent a penalty associated with each of the possible estimates. In this section we provide the Bayes estimates for the parameter θ based on three loss functions.

3.1 Bayes Estimator and Posterior Risks under Quasi-Quadratic Loss Function (QQLF)

The quasi-quadratic loss function is of the form:

$$L\left(\theta,\hat{\theta}\right)=\left(e^{-c\hat{\theta}}-e^{-c\theta}\right)^2,\qquad c\neq 0.$$

The Bayes estimator and the posterior risk under QQLF are given below:

$$\hat{\theta}_{QQLF}=\frac{-1}{c}\ln\left\{E\left(e^{-c\theta}\middle|x\right)\right\},\quad \rho\left(\hat{\theta}_{QQLF}\right)=E\left(e^{-c\theta}\right)-\left[E\left(e^{-c\theta}\right)\right]^2.$$

For simulation study we consider $c=1$.

The Bayes estimator and posterior risks under this loss function are:

$$\hat{\theta}=\log\left\{\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{1}{\left(1+\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{1}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}\right\},$$

$$\rho\left(\hat{\theta}\right)=\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{1}{\left(2+\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{1}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}-$$

$$-\left[\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{1}{\left(1+\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{1}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}\right].$$

3.2 Bayes Estimator and Posterior Risks under Squared-Log Error Loss Function (SLELF)

The squared-log error loss function is of the form:

$$L\left(\theta,\hat{\theta}\right)=\left(\ln\hat{\theta}-\ln\theta\right)^2,$$

which is balanced with $\lim L\left(\theta,\hat{\theta}\right)\rightarrow\infty$ as $\hat{\theta}\rightarrow 0$ or ∞ A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function only considers error of estimation. This loss function is convex for $\frac{\hat{\theta}}{\theta}\leq e$ and concave otherwise, but its risk function has a unique minimum with respect to $\hat{\theta}$. The

Bayes estimator for the parameter θ of Burr Type-II distribution under the squared-log error loss function may be given as:

$$\hat{\theta}_{SLELF} = \exp\left\{E\left(\ln\theta|\,x\right)\right\}, \quad \rho\left(\hat{\theta}_{SLELF}\right) = E\left[\left(\ln\theta|\,x\right)\right]^2 - \left[E\left(\ln\theta|\,x\right)\right]^2,$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and

$$\psi'(x) = \frac{d^2}{dx^2}\left\{\log\Gamma(x)\right\} = \frac{d}{dx}\left\{\frac{\Gamma'(x)}{\Gamma(x)}\right\}$$

is the tri-gamma function.

The Bayes estimator and posterior risks SLELF are:

$$\hat{\theta} = \frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}\left\{\frac{\exp\{\psi(R+1-j+m(a-1))\}}{\psi_{i,r,k}+lw+mb}\right\}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}.$$

$$\rho\left(\hat{\theta}\right) = \frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}\psi'\left[R+1-j+m\left(a-1\right)\right]}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{\left(\psi_{i,r,k}+lw+mb\right)^{R+1-j+m(a-1)}}}.$$

3.3 Bayes estimator and posterior risks under precautionary loss function (PLF)

Norstrom [20] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions as a special case. These estimators are very useful when underestimation may lead to serious consequences. A very useful and asymmetric precautionary loss function is

$$L_{PLF}\left(\hat{\theta},\theta\right) = \frac{\left(\theta-\hat{\theta}\right)^2}{\theta}$$

The Bayes estimator and the posterior risk under Precautionary loss function are given below:

$$\hat{\theta}_{PLF} = \sqrt{E_{\phi/x}\left(\theta^2\right)},\ \rho\left(\hat{\theta}_{PLF}\right) = 2\{\sqrt{E_{\phi/x}\left(\theta^2\right)} - E_{\phi/x}\left(\theta\right)\}.$$

The Bayes estimator and Posterior risks under PLF are:

$$\hat{\theta} = \sqrt{\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+3-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+3-j+m(a-1)}}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+1-j+m(a-1)}}}}$$

$$\rho\left(\hat{\theta}\right) = 2\left[\sqrt{\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+3-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+3-j+m(a-1)}}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+1-j+m(a-1)}}}}\right] -$$

$$-\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+2-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+2-j+m(a-1)}}}{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+1-j+m(a-1)}}}$$

4 Elicitation

Elicitation is the process of talking out the expert knowledge about some unknown quantity of interest, or the probability of some future event, which can be used to supplement any numerical data we may have. If the expert in question does not have a statistical background, as often happens, translating their beliefs into a statistical form suitable for the use in our analyses can be a challenging task as described Dey [9].

Prior elicitation is an organized and systematic approach to represent an expert's opinion as a well-defined, coherent prior. In Bayesian analysis, specification and elicitation of the prior distribution is a common difficulty. The Bayesian approach allows the use of objective data or subjective opinion in specifying a prior distribution. Elicitation is the process of extracting experts' knowledge about some parameter of interest, or the probability of some future event and also the quantification of this prior information accurately, which then supplements the given data. In any statistical analysis there will typically be some form of background knowledge available in addition to data at hand. Berger [7] gives a description of numerous different methods for the elicitation of prior distribution. For different sampling models, different methods for specification of opinions have been developed. There are various methods of elicitation available in literature (reader desires more detail see Grimshaw et al. [13], Kadane [14], O'Hagan et al. [19], Kadane et al. [15], Jenkinson [16] and Leon et al. [17]. Here we use the method based on the prior predictive distribution, which is developed by Aslam [6].

4.1 Elicitation of Hyperparameter

Bayesian analysis elicitation of opinion is a crucial step. It helps to make it easy for us to understand what the experts believe in and what their opinions are. In statistical inference the characteristics of a certain predictive distribution proposed by an expert determine the hyperparameters of a prior distribution.

In this article, we focus on a probability elicitation method known as prior predictive elicitation. Predictive elicitation is a method for estimating hyperparameters of prior distributions by inverting corresponding prior predictive distributions. Elicitation of hyperparameter from the prior $p(\theta)$ is conceptually difficult task because we first have to identify prior distribution and then its hyperparameters. The prior predictive distribution is used for the elicitation of the hyperparameters which is compared with the experts' judgment about this distribution and then the hyperparameters are chosen in such a way so as to make the judgment agree closely as possible with the given distribution. According to Aslam [6], the method of assessment is to compare the predictive distribution with experts' assessment about this distribution and then to choose the hyperparameters that make the assessment agree closely with the member of the family. He discusses three important methods to elicit the hyperparameters: (i) Via the

Prior Predictive Probabilities (ii) Via Elicitation of the Confidence Levels (iii) Via the Predictive Mode and Confidence Level.

4.2 Prior Predictive Distribution

The prior predictive distribution is:

$$p(y) = \int_0^\infty p(y/\theta)\, p(\theta)\, d\theta.$$

The predictive distribution under exponential prior is:

$$p(y) = \int_0^\infty \theta e^{-y} \left(1 + e^{-y}\right)^{-(\theta+1)} w e^{-\phi w} d\theta.$$

After some simplification it reduces as

$$p(y) = \frac{w e^{-y}}{(1 + e^{-y})\{w + \ln(1 + e^{-y})\}}, \quad -\infty < y < \infty.$$

The predictive distribution under the Gamma prior is:

$$p(y) = \frac{(a-1)\, b^{a-1} e^{-y}}{(1 + e^{-y})\{b + \ln(1 + e^{-y})\}^a},$$

By using the method of elicitation defined by Aslam [6], we obtain the following hyper parameters $w = 3.65943$, $a = 2.20763$ and $b = 3.09598$.

5 Credible and Posterior Predictive Intervals

The posterior predictive distribution of the future observation y is:

$$p(y/x) = \int_0^\infty p(y/\theta)\, p(\theta/x)\, d\theta,$$

where $p(\theta/x)$ is the posterior distribution under the respective prior and $p(y/\theta)$ is the data model described in section 1.

The $(1-\alpha)\,100\%$ Bayesian Predictive Interval (L, U) is obtained by solving the two equations

$$\int_{-\infty}^{L} p(y/x)\, dy = \frac{\alpha}{2} = \int_U^\infty p(y/x)\, dy.$$

On simplification these equations, predictive intervals are obtained.

The generalized posterior predictive distribution of the future observation y is:

$$p\left(y/x\right)=\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\frac{e^{-y}}{(1+e^{-y})}\left[\frac{R+1-j+m(a-1)}{\{\psi_{irk}+lw+mb+\ln(1+e^{-y})\}^{R+2-j+m(a-1)}}\right]}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left(\psi_{irk}+lw+mb\right)^{-\{R+1-j+m(a-1)\}}},$$

and the predictive interval is:

$$\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\{\psi_{irk}+lw+mb+\ln\left(1+e^{-L}\right)\}^{-\{R+1-j+m(a-1)\}}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left(\psi_{irk}+lw+mb\right)^{-\{R+1-j+m(a-1)\}}}=\frac{\alpha}{2}$$

$$\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left(\psi_{irk}+lw+mb\right)^{-\{R+1-j+m(a-1)\}}}\Big\{\left(\psi_{irk}+lw+mb\right)^{-\{R+1-j+m(a-1)\}}$$
$$-\{\psi_{irk}+lw+mb+\ln\left(1+e^{-L}\right)\}^{-\{R+1-j+m(a-1)\}}\Big\}=\frac{\alpha}{2}$$

According to Eberly and Casella [10] the credible interval can be defined as:

$$\int_{0}^{L}g\left(\theta/x\right)d\theta=\frac{\alpha}{2},\quad\int_{U}^{\infty}g\left(\theta/x\right)d\theta=\frac{\alpha}{2},$$

where L and U are the lower and upper limits of the credible interval respectively and is the level of significance.

The $(1-\alpha)\,\%$ credible interval for θ has been derived as:

$$\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1),L\{\psi_{irk}+lw+mb\})}{(\psi_{irk}+lw+mb)^{R+1-j+m(a-1)}}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+1-j+m(a-1)}}}=1-\frac{\alpha}{2}$$

$$\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1),U\{\psi_{irk}+lw+mb\})}{(\psi_{irk}+lw+mb)^{R+1-j+m(a-1)}}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\frac{\Gamma(R+1-j+m(a-1))}{(\psi_{irk}+lw+mb)^{R+1-j+m(a-1)}}}=\frac{\alpha}{2}$$

The close form expressions for the limits of posterior predictive and credible intervals are not possible, so these have been evaluated numerically.

6 Simulation Study

Monte Carlo simulation techniques are widely used in statistical research. Since real-world data sets can often be radically non-normal, it is essential that statisticians have a variety of techniques available for univariate or multivariate non-normal data generation. This section shows how simulation can be helpful and illuminating way to approach problems in Bayesian analysis. Bayesian problems of updating estimates can be handled easily and straight forwardly with simulation. Here, the inverse transformation method of simulation is used to compare the performance of different estimators. The study has been carried out for different values of (n, r) using $\theta \in (1, 2 \text{ and } 4)$. Censoring rate is assumed to be 20%. Sample size is varied to observe the effect of small and large samples on the estimators. Changes in the estimators and their risks have been determined when changing the loss function and the prior distribution of while keeping the sample size fixed. All these results are based on 1,000 repetitions. In the tables, the estimators for the parameter and the risk, is averaged over the total number of repetitions. Mathematica 8.0 has been used to carry out the results. The results are summarized in the following Tables.

Table 1: Bayes Estimates and the Posterior Risks (given in parentheses) under QQLF.

	Uniform			Jeffreys		
n	$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 1$	$\theta = 2$	$\theta = 4$
20	1.08337	2.12322	4.04865	1.01443	1.95923	3.81435
$r = 3$, $n - s = 18$	(0.006580)	(0.003639)	(0.000547)	(0.006958)	(0.004373)	(0.000747)
40	1.05216	2.06496	4.02563	1.01290	1.97447	3.89022
$r = 5$, $n - s = 36$	(0.003505)	(0.001912)	(0.000218)	(0.003620)	(0.002129)	(0.000260)
60	1.02975	2.0306	4.01376	1.01348	1.98996	3.93487
$r = 7$, $n - s = 54$	(0.002395)	(0.001317)	(0.000132)	(0.002438)	(0.001392)	(0.000149)
80	1.02634	2.01130	4.00768	1.01042	2.00628	3.94771
$r = 9$, $n - s = 72$	(0.001441)	(0.000991)	(0.000094)	(0.002042)	(0.001001)	(0.000102)

Table 2: Bayes Estimates and the Posterior Risks (given in parentheses) under QQLF.

	Gamma			Exponential		
n	$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 1$	$\theta = 2$	$\theta = 4$
20	0.97708	1.64457	2.54737	0.88501	1.48056	2.24324
$r = 3$, $n - s = 18$	(0.006299)	(0.004893)	(0.002141)	(0.006620)	(0.005776)	(0.003091)
40	0.98616	1.81058	3.08058	0.93687	1.71147	2.85086
$r = 5$, $n - s = 36$	(0.003417)	(0.002298)	(0.000596)	(0.003523)	(0.002565)	(0.000805)
60	0.98888	1.86267	3.32674	0.95693	1.79441	3.14724
$r = 7$, $n - s = 54$	(0.002344)	(0.001495)	(0.000291)	(0.002394)	(0.001617)	(0.000370)
80	0.99985	1.88726	3.47249	0.97492	1.85865	3.32477
$r = 9$, $n - s = 72$	(0.001851)	(0.001118)	(0.000177)	(0.001634)	(0.001077)	(0.000205)

Table 3: Bayes Estimates and the Posterior Risks (given in parentheses) under SLELF.

	Uniform			Jeffreys		
n	$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 1$	$\theta = 2$	$\theta = 4$
20	1.10071	2.17676	4.30386	1.02535	2.06293	4.12472
$r = 3$, $n - s = 18$	(0.060588)	(0.060588)	(0.060588)	(0.064494)	(0.064494)	(0.064494)
40	1.04721	2.09115	4.17613	1.01281	2.04231	4.05808
$r = 5$, $n - s = 36$	(0.030767)	(0.030767)	(0.030767)	(0.031743)	(0.031743)	(0.031743)
60	1.03319	2.04943	4.10180	1.01130	2.02058	4.03848
$r = 7$, $n - s = 54$	(0.020618)	(0.020618)	(0.020618)	(0.021052)	(0.021052)	(0.021052)
80	1.01956	2.04614	4.05904	1.01737	2.00386	4.02763
$r = 9$, $n - s = 72$	(0.015504)	(0.015504)	(0.015504)	(0.015751)	(0.015751)	(0.015751)

Table 4: Bayes Estimates and the Posterior Risks (given in parentheses) under SLELF.

	Gamma			Exponential		
n	$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 1$	$\theta = 2$	$\theta = 4$
20	0.97011	1.68530	2.64059	0.88127	1.51125	2.29533
$r = 3$, $n - s = 18$	(0.056458)	(0.056458)	(0.056458)	(0.060588)	(0.060588)	(0.060588)
40	0.98476	1.83429	3.16371	0.94424	1.69771	2.92610
$r = 5$, $n - s = 36$	(0.029665)	(0.029665)	(0.029665)	(0.030767)	(0.030767)	(0.030767)
60	0.99212	1.88327	3.37067	0.95993	1.80120	3.19760
$r = 7$, $n - s = 54$	(0.020117)	(0.020117)	(0.020117)	(0.020618)	(0.020618)	(0.020618)
80	0.99511	1.91491	3.53574	1.96830	1.85793	3.35824
$r = 9$, $n - s = 72$	(0.015218)	(0.015218)	(0.015218)	(0.015506)	(0.015506)	(0.015506)

Table 5: Bayes Estimates and the Posterior Risks (given in parentheses) under PLF.

	Uniform			Jeffreys		
n	$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 1$	$\theta = 2$	$\theta = 4$
20	1.14368	2.28236	4.52336	1.09210	2.16090	4.34216
$r = 3$, $n - s = 18$	(0.057935)	(0.115626)	(0.229140)	(0.058374)	(0.115301)	(0.23142)
40	1.05932	2.18107	4.30268	1.03921	2.08901	4.18807
$r = 5$, $n - s = 36$	(0.028068)	(0.057791)	(0.114005)	(0.028293)	(0.056857)	(0.113987)
60	1.04795	2.11418	4.19889	1.02329	2.05645	4.13339
$r = 7$, $n - s = 54$	(0.018801)	(0.037929)	(0.075330)	(0.018694)	(0.037567)	(0.074781)
80	1.03258	2.06682	4.14805	1.01930	2.03417	4.07619
$r = 9$, $n - s = 72$	(0.013919)	(0.028055)	(0.056563)	(0.013443)	(0.027945)	(0.056049)

Table 6: Bayes Estimates and the Posterior Risks (given in parentheses) under PLF.

	Gamma			Exponential		
n	$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 1$	$\theta = 2$	$\theta = 4$
20	1.03241	1.75991	2.76937	0.93927	1.58680	2.42453
$r = 3$, $n - s = 18$	(0.049280)	(0.084001)	(0.132176)	(0.047576)	(0.080369)	(0.122792)
40	1.01595	1.87127	3.26288	0.97113	1.75711	3.01476
$r = 5$, $n - s = 36$	(0.026084)	(0.048043)	(0.083767)	(0.083767)	(0.046554)	(0.079872)
60	1.01414	1.89932	3.46487	0.98349	1.83841	3.29137
$r = 7$, $n - s = 54$	(0.017808)	(0.033351)	(0.060840)	(0.017644)	(0.032981)	(0.059045)
80	1.00758	1.93489	3.58552	1.97772	1.86817	3.41486
$r = 9$, $n - s = 72$	(0.013485)	(0.025552)	(0.047567)	(0.013310)	(0.024298)	(0.044395)

Table 7: The lower limit (LL), the upper limit (UL) and the width of the 95% Credible Intervals under Uniform.

	$\theta = 1$			$\theta = 2$			$\theta = 4$		
r, n, $n - s$	LL	UL	Width	LL	UL	Width	LL	UL	Width
3, 20, 18	0.63646	1.58291	0.94645	1.28008	3.18362	1.90354	2.59301	6.45160	3.85859
5, 40, 36	0.71975	1.37702	0.65727	1.45683	2.78720	1.33037	2.91498	5.57692	2.66194
7, 60, 54	0.76271	1.29703	0.53432	1.55013	2.63611	1.08598	3.07932	5.2366	2.15728
9, 80, 72	0.79749	1.26368	0.46619	1.58594	2.51312	0.92718	3.19365	5.06071	1.86706

Table 8: The lower limit (LL), the upper limit (UL) and the width of the 95% Credible Intervals under Jeffreys.

	$\theta = 1$			$\theta = 2$			$\theta = 4$		
r, n, $n - s$	LL	UL	Width	LL	UL	Width	LL	UL	Width
3, 20, 18	0.59354	1.51450	0.92096	1.19376	3.04604	1.85228	2.41805	6.17246	3.75441
5, 40, 36	0.69661	1.34489	0.64828	1.41000	2.72217	1.31217	2.82127	5.44680	2.62553
7, 60, 54	0.74674	1.27617	0.52943	1.51769	2.59371	1.07602	3.01487	5.15238	2.13751
9, 80, 72	0.78518	1.24814	0.46296	1.56144	2.48221	0.92077	3.14406	4.99846	1.85440

Table 9: The lower limit (LL), the upper limit (UL) and the width of the 95% Credible Intervals under Exponential.

	$\theta = 1$			$\theta = 2$			$\theta = 4$		
r, n, $n - s$	LL	UL	Width	LL	UL	Width	LL	UL	Width
3, 20, 18	0.52879	1.31508	0.78629	0.90816	2.25854	1.35038	1.41701	3.52441	2.10740
5, 40, 36	0.65365	1.25056	0.59691	1.20933	2.31365	1.10432	2.0681	3.95658	1.88948
7, 60, 54	0.71457	1.21516	0.50059	1.36345	2.31862	0.95517	2.42086	4.11680	1.69594
9, 80, 72	0.75876	1.20236	0.44360	1.44003	2.28167	0.84164	2.64371	4.20233	1.55862

Table 10: The lower limit (LL), the upper limit (UL) and the width of the 95% Credible Intervals under Gamma.

	$\theta = 1$			$\theta = 2$			$\theta = 4$		
r, n, $n-s$	LL	UL	Width	LL	UL	Width	LL	UL	Width
3, 20, 18	0.58752	1.42035	0.83283	1.02876	2.48703	1.45827	1.64849	3.98577	2.33728
5, 40, 36	0.68884	1.30416	0.61532	1.29017	2.44260	1.15243	2.24923	4.25832	2.00909
7, 60, 54	0.73984	1.25089	0.51105	1.42437	2.40826	0.98389	2.56664	4.33953	1.77289
9, 80, 72	0.77869	1.22939	0.45070	1.48806	2.34884	0.86078	2.77376	4.37906	1.60530

7 Estimation under Real Life Data Set

In this section, we analyze a real data set and illustrate the analysis of the posterior distribution of shape parameter of Bur type-II assuming informative and non-informative priors. The data set is taken from Lawless [18] and it represent breaking strengths of single carbon fibers of length 10. The sample characteristics required to evaluate the estimates of shape parameter of Burr-type II are as follows: $n = 60$, $r = 7$, $s = 54$ and $\sum_{i=1}^{60} \ln(1 + e^{-x_i}) = 2.68480$.

7.1 Graphical Results of Posterior Distribution for Real Life Data Set

The below graphs reveal that posterior distributions under different informative and non informative priors are slightly positively skewed.

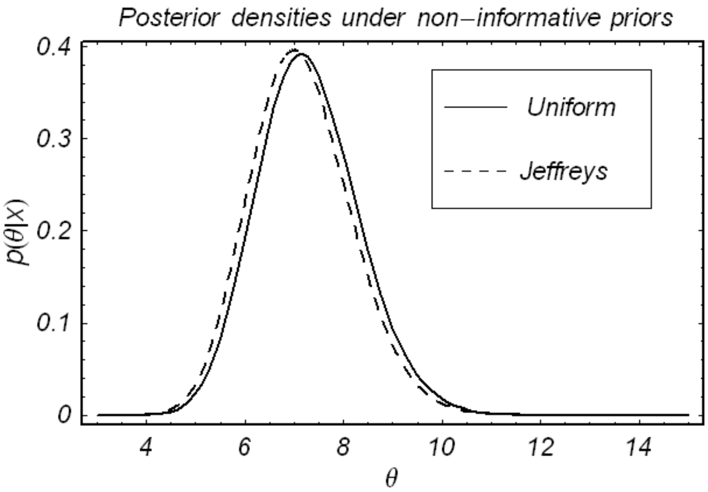

Figure 1: Posterior densities under Uniform and Jeffreys

Table 11: Bayes Estimates and the Posterior Risks using the real Data Set.

	QQLF		SLELF		PLF	
Prior	BEs	PRs			BEs	PRs
Uniform	6.80948	1.5097×10^{-6}	7.21633	0.02062	7.36274	0.144293
Jeffreys	6.67487	1.9159×10^{-6}	7.07148	0.02105	7.21782	0.144083
Exponential	4.58198	4.6903×10^{-5}	4.74122	0.02062	4.83598	0.09192
Gamma	4.93422	2.6998×10^{-5}	5.12029	0.02012	5.22048	0.097546

8 Conclusion

The simulation study has displayed some interesting properties of the Bayes estimates. The risks of the estimates seem to be large in case when the value of the parameter is large and small for relative smaller value of the parameter except under quasi-quadratic loss function. However, the risks under said loss functions are reduced as the sample size increases. Another interesting remark concerning the risks of the estimates is that increasing (decreasing) the value of the parameter reduces (increases) the risks of the estimates under quasi-quadratic loss function. The performance of squared-log error loss function is independent of choice of parametric value. The effect of the increasing values of the parameter is in the form of underestimation assuming informative priors. In comparison of non-informative priors the uniform prior provides the better estimates as the corresponding risks are smaller under quasi-quadratic and squared-log error loss functions. On the other hand, the Jeffreys prior provides the better estimates under precautionary loss function with few exceptions. Further, in making comparison

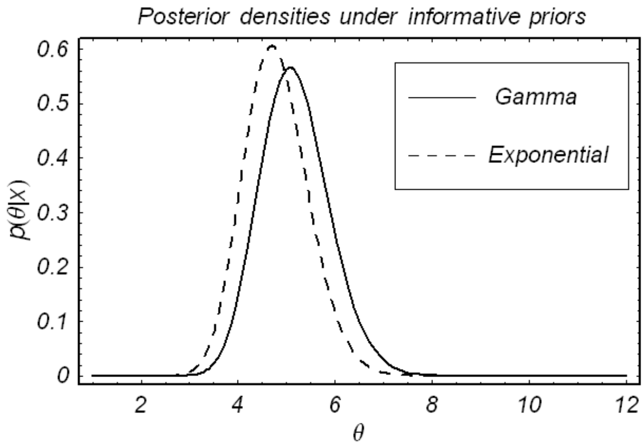

Figure 2: Posterior densities under Gamma and Exponential

of informative priors the gamma prior under quasi-quadratic and squared-log error loss functions gives the best results. While the exponential prior turns out to perform better under precautionary loss function, therefore it produces more efficient estimates as compared to the other priors. In addition, the estimates under quasi-quadratic loss function give the minimum risks among all loss functions for each prior. It can also be observed that the performance of estimates under informative priors is better than those under non-informative priors. The credible intervals are in accordance with the point estimates, that is, the width of credible interval is inversely proportional to sample size while, it is directly proportional to the parametric value. The credible interval assuming exponential prior are much narrower than the credible intervals assuming non-informative prior. It is the use of prior information that makes a difference in terms of gain in precision. The results from the analysis of real life data are compatible with the simulation study.

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