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Efficiency of bayesian heteroscedastic linear model

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In order to investigate the asymptotic efficiency of estimators under two different simulation techniques, normal-normal double sided Heteroscedastic error structure was adopted. We explored Direct Monte Carlo method of Zellner et al. (2010) and Metropolis Hasting Algorithm experiments, an approach of Markov Chain Monte Carlo.

We truncated the model with one error component of two sided error structure. A Metropolis-Hasting Algorithm and Direct Monte Carlo adopted to perform simulation on marginal posterior distribution of heteroscedastic linear econometric model. Since Ordinary Least squares is invalid and inefficient in the presence of heteroscedastic, heteroscedastic linear model was conjugated with informative priors to form posterior distribution. Maximum Likelihood Estimation was compared with Bayesian Maximum Likelihood Estimation, Mean Squares Error criterion was use to identify which estimator and/or simulation method outperform other. We chose the following sample sizes: 25; 50; 100; and 200. Thus 10,000 simulations with varying degree of heteroscedastic error structures were adopted. This is subject to the level of convergence.

In the overall, minimum mean squares error criterion revealed improving performance asymptotically regardless of the degree of heteroscedasticity. The results showed that Direct Monte Carlo Method outperformed Markov Chain Monte Carlo Method and Maximum Likelihood Estimator with minimum mean square error at any degree of heteroscedasticity.

keywords: Markov Chain Monte Carlo Method, Heteroscedasticity, Bayesian Maximum Likelihood Estimator, Metropolis-Hasting Algorithm, Direct Monte Carlo Method.

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1 Introduction

Recently, numerous literatures emerged in the field of Bayesian Statistics; this is due to the effort of people like George Casella and Christian Robert, Jim Albert, Reuven Rubinstein, Dirk Kroese and other numerous researchers who brought into limelight the simulation techniques of Markov Chain Monte Carlo technique into the field of Bayesian Statistics in the early 90s. Fewer papers were published in the area of Bayesian heteroscedastic model, chiefly amongst are the works of Arto et al. (2008) in their paper Bayesian two-stage regression with parametric heteroscedasticity where they allowed for unequal variances, Carlos et al. (2012) in their paper the multiplicative heteroscedastic von Bertalanlfy model.

Most often economic problems rely on regression model which in most cases come with errors of which the most significant of it all is heteroscedasticity of either one or two components which may be additive or multiplicative (Oloyede, 2010). The consequence of heteroscedasticity for Ordinary least squares estimation is quite serious. Estimator remains unbiased, but is no longer efficient. More importantly, the standard errors usually computed for the least squares estimators are no longer appropriate, and hence confidence intervals and hypothesis tests that use these standard errors are invalid (Hadri et al., 1999).

Obviously, when heteroscedasticity exists ignoring it may lead to substantial biases and even inconsistent estimates, but correcting for it leads not only to a substantial improvement of the statistical properties of estimators but also to improved efficiency (Hadri et al., 1999).

Econometrics models are usually expressed in terms of an unknown vector of parameters $\underline{\theta}\epsilon\Theta \subseteq R^k$, which fully specifies the joint probability distribution of the observations $X = (x_1, , x_T)$. Bayesian inference proceeds from the likelihood function and prior information usually expressed as a probability density function over the parameters, $(\underline{\theta})$, it being implicit that $\pi(\underline{\theta})$ depends on the conditioning set of prior information. The posterior distribution is proportional to $p(\underline{\theta}) \propto \pi(\underline{\theta})L(\underline{\theta})$ (John , 1989).

John (2005) used Monte Carlo integration to estimate the parameters and asserted that diffuse priors may incorporate inequality restriction which arise frequently in applied work but are impractical if not impossible to handle in classical setting. Carlos et al. (2012) considered a multiplicative heteroscedastic dispersion matrix. All estimates were obtained using a sampling based approach, which allows information to be input before hand with lower computational effort.

Ignoring heteroscedasticity disturbances in econometric models does not in general prevent consistent point estimation, or even consistent interval estimation and hypothesis tests (Whites, 1980), but it typically entails inefficient point estimators and hypothesis tests with suboptimal asymptotic local power (Robinson., 1987).

Unless the form of heteroscedasticity is of interest in itself, it may be better to avoid attempting to parameterize it , and methods have been proposed that do not claim to account optimally for a particular form of heteroscedasticity , but have good efficiency properties with respect to heteroscedasticity of at best loosely defined form Robinson. (1987). Since the efficiency of parameters from the model depends on the nature of the error variance estimator, a great deal of effort has been put in it to develop techniques for obtaining consistent estimates of each residual variance in linear models when heteroscedasticity is suspected (Senyo, 1993). Surekha et al. (1984) observed that the efficiency of the EGLS estimator rests more on the choice of estimator and sample size rather than on the specification of the correct residual variance structure.

Goldfeld et al. (1972) tried different specifications of heteroscedastic error structures, and from Monte Carlo Study results concluded that the overall performances of all other competing estimators was observed to be somewhat sensitive to the heteroscedastic error structure. Mackinnon et al. (1985) also used sampling experiments to show that, in a lot of situations, traditionally, heteroscedasticityconsistent covariance matrix estimators can sometimes produce more grossly misleading results that the usual ordinary least squares covariance matrix estimator that neglects the presence of heteroscedasticity .

The close similarly between the performance of heteroscedastic error model estimators based on additive error structure and those based on exponential family error structure is not surprising. This is so because the exponential error structure specification encompasses the additive specification. It can, therefore, be said that the better performance of heteroscedastic error model estimators based on the exponential error structure is due to its ability to approximate in a better fashion the true unknown heteroscedastic error structure than any other structure. Thus, its encompassing ability makes it a more general approximation of what the true unknown underlying heteroscedastic error structure may be (Senyo, 1993).

Senyo (1993) recommended that when true error structure is not known a priori, the applied researcher should hypothesize and use exponential heteroscedastic error structures in order to obtain more efficient parameters of the model. One of the simulation approaches that does not suffer from the computational problems of MCMC method is a direct Monte Carlo procedure (Zellner et al., 2002).

In this paper, multiplicative double sided error structure with one component was incorporated. Many attempts have been made in the literature to look into the efficiency of estimators in linear model using Monte Carlo simulation, but we endaevour to carry out simulation using MCMC and DMC Simulation methods; this paper also examined the efficiency of the estimators from small sample to large sample. These are the gaps, this study decides to fill.

2 Model Designs

Let $y = X\beta + u$ with $u \sim [N(0, \sigma_i^2 \Omega)]$ where Ω is a positive definite matrix of order n .A case where $u \sim [N(0, \sigma^2 I)]$ is a homoscedastic model with constant variance, but when $u \sim [N(0, \sigma_i^2 \Omega)]$ indicates unequal variances of the diagonal element of matrix $n \times n$ which is regarded as heteroscedastic error structure.

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + u \tag{1}$$

Let X denote X_1 and X_2 with multiplicative heteroscedasticity using Harvey. (1976) which can be expressed as $\sigma_i^2 = \sigma^2 exp(\beta_1 X_1 + \beta_2 X_2)^\delta$ where δ is an unknown parameter which determine the degree of heteroscedasticity. Adopting a full Bayesian inference, we examine the likelihood function, prior distribution for the parameters and hyper-parameters in the model and with MCMC algorithm. The likelihood function of $\theta$and σ, where $\theta = (\beta_0, \beta_1, \beta_2, \delta)$give the sample vector $X_1,\ X_2 = (1, 2, , n)'$ and $y = (y_1,\ y_2, , y_n)'$ is expressed as

$$L\left(\theta, \sigma|X, y\right) = \left(2\pi\sigma^2\right)^{-n/2} \prod_{i=1}^{n} exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[y_i - x\beta\right]^2\right\} \tag{2}$$

Incorporating multiplicative heteroscedastic into our likelihood estimator we derived from the product of the error density function.

$$L\left(\theta, \sigma|X, y\right) = \left(2\pi\sigma^2\right)^{-n/2} \prod_{i=1}^{n} w^{-\delta/2} exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} w^{-\delta}\left[y_i - x\beta\right]^2\right\} \tag{3}$$

To derive the full Bayesian density, we truncate the error density function eq 3 with multinomial distribution density. Marginal posterior density is obtained by integrating the joint posterior density with respect to each parameter, thus, expert opinion can be adopted by assuming the set of parameters $\beta_0, \beta_1, \beta_2, \delta\ and\ \sigma$ as independent marginal distribution.

We assumed a prior density. $\pi\left(\beta_0, \beta_1, \beta_2, \delta, \sigma\right) = \pi\left(\beta_0\right)\pi(\beta_1)\pi(\beta_2)\,\pi\left(\delta\right)\pi\left(\sigma\right)$. Thus multivariate normal distribution is considered for,β while inverse gamma is considered for σ and a uniform distribution is considered for δsuch that

$$\pi\left(\beta\right) \propto \left(2\pi\sigma^2\right)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\ \left(\beta - \mu\right)^2\right\},\ \beta > 0 \tag{4}$$

$$\pi(\sigma^2) \propto \left(\sigma^2\right)^{-a_1+1} \exp\left(-b_1/\sigma^2\right),\ \sigma^2 > 0 \tag{5}$$

$$\pi\left(\delta\right) \propto c \tag{6}$$

c is constant

The posterior distribution of $\theta = (\beta_0, \beta_1, \beta_2, \delta, \sigma)$ considering independence among the parameters is given by :

$$\pi\left(\beta_0, \beta_1, \beta_2, \delta, \sigma|X, y\right) \propto \left(2\pi\sigma^2\right)^{-\frac{n}{2}} \left(\sigma^2\right)^{-(a_1-1-n/2)} \exp\left\{-\frac{1}{2\sigma^2}(\beta - \mu)^2\right\} \prod_{i=1}^{n} w-$$

$$-\delta/2exp\{-\frac{1}{\sigma^4}(b_1 + \frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2\} \tag{7}$$

where $a_1,\ \ b_1$ are the hyper-parameters for the inverse-gamma distribution. Hyper-parameters are excluded for β-parameters since they would be estimated from the data and may be arbitrarily small leading to problems which may eventually affect the inferences. Integrating the posterior $\pi(\beta, \delta, \sigma|X, y)\ \sigma$, thus we have joint a posterior distribution for (β, δ)

$$\pi(\beta_0, \beta_1, \beta_2, \delta, \sigma|X, y) \propto (2\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\ (\beta-\mu)^2\right\} \prod_{i=1}^{n} w{-}\delta/2exp\{-b_1 - \frac{1}{2}$$

$$\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2\}^{-(a_{1-n/2})} \tag{8}$$

Metropolis Hasting Algorithm update is performed on the full conditional distribution of $\sigma^2 \propto IG(a_1+\frac{n}{2},\ b_1+\ \frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2)$. This yields the following full conditional density of the parameters β and:σ

$$\pi(\beta_0|\delta, X, y) \propto\ \exp\left\{-\frac{1}{2}\ (\beta_0-\mu)^2\right\} \prod_{i=1}^{n} w^{-\delta/2}exp\{-\frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2\}^{-(a_{1-n/2})} \tag{9}$$

$$\pi(\beta_1|\delta, X, y) \propto\ \exp\left\{-\frac{1}{2}\ (\beta_1-\mu)^2\right\} \prod_{i=1}^{n} w^{-\delta/2}exp\{-\frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2\}^{-(a_{1-n/2})} \tag{10}$$

$$\pi(\beta_2|\delta, X, y) \propto\ \exp\left\{-\frac{1}{2}\ (\beta_2-\mu)^2\right\} \prod_{i=1}^{n} w^{-\delta/2}exp\{-\frac{1}{2}\sum_{i=1}^{n} w \tag{11}$$

$$\pi(\sigma|\theta.X, y) \propto \left(\sigma^2\right)^{-(a_1-1-n/2)} exp\left(-b_1/\sigma^2\right) \prod_{i=1}^{n} w^{-\delta/2}exp\{-\frac{1}{\sigma^2}(b_4 + \frac{1}{2}\sum_{i=1}^{n} w$$

$$-\delta(y - X\beta)^2\}^{-(a_{1+n/2})} \tag{12}$$

$$\pi(\delta|\beta, X, y) \propto \prod_{i=1}^{n} w^{-\delta/2}(b_1 + \frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2\}^{-(a_{4+n/2})} \tag{13}$$

2.1 Metropolis-Hasting Algorithm

The Metropolis algorithm Metropolis et al. (1949), Metropolis et al. (1953) generates a sequence of selections from this distribution is as follows:

1. Start with any initial value θ_0 satisfying $(\theta_0) > 0$.
2. Using current θvalue, sample a candidate point θ^* from some jumping distribution, $q(\theta_1, \theta_2)$ which is the probability of returning a value of given a previous value of. This distribution is also referred to as the proposal or candidate-generating

distribution. The only restriction on the jump density in the Metropolis algorithm is that it is symmetric, i.e.

$$q(\theta_1, \theta_2) = q(\theta_2, \theta_1)$$

3. Given the candidate point,θ^* calculate the ratio of the density at the candidate (θ^*) and current (θ_{t-1}) points, $\alpha = \frac{p(\theta^*)}{p(\theta_{t-1})} = \frac{f(\theta^*)}{f(\theta_{t-1})}$ Note: we are considering the ratio of $p(x)$ under two different values, the normalizing constant K cancels out.

4. If the jump increases the density,($\alpha > 1$) accept the candidate point ($set\ \theta_t = \theta^*$) and return to step 2. If the jump decreases the density ($\alpha < 1$) then with probability accept the candidate point, else reject it and return to step 2.

We can summarize the Metropolis sampling as first computing $\alpha = min\left(\frac{f(\theta^*)}{f(\theta_{t-1})}, 1\right)$ and then accepting a candidate point with probability α(the probability of a move). This generates a Markov chain $(\theta_1, \theta_2, , \theta_k, \ldots)$, as the transition probabilities from θ_t to θ_{t+1} depends only on θ_t and not $(\theta_0, , \theta_{t-1})$. Following a sufficient burn-in period (of, say, $k$steps), the chain approaches its stationary distribution and , samples from the vector $(\theta_{k+1}, , \theta_{k+n})$ are samples from.$p(x)$

2.2 A direct Monte Carlo Sampling Procedure

The direct Monte Carlo procedure can be repeated many times to yield draws from the joint posterior density in the following way. We can draw σ_i^2 from the inverse gamma density $\pi\left(\sigma_i^2|D\right)$ and insert the drawn value in $\pi\left(b|\sigma_i^2, D\right)$ and make a draw from it. This procedure is applied for $j = 1, , m$This procedure is then repeated many times. The algorithm is summarized as follows:

1. (Initialization) Set the number of samples $N$to be generated. Set $j = m$.

2. Generate σ_i^2, $k = 1, , N$, and insert the drawn values in.$\pi\left(b|\sigma_i^2, D\right)$ Then make a draw from, for.

3. Increase the index by one. Draw from the conditional inverse gamma density and then generate from,

4. Repeat Step 2 sequentially until.

3 Data Generation Processes

In an attempt to investigate the asymptotic efficiency/performance of estimators of econometric model in the presence of heteroscedastic error structure, we adopted Markov Chain Monte Carlo and Direct Monte Carlo Experiments. The sample sizes are specified with 4 sets as follow: 25, 50, 100 and 200 .Harvey. (1976) multiplicative heteroscedastic error structure was adopted to truncate linear econometric model. The scale of $\delta-$

heteroscedastic error structure is selected as 0.0 homoscedastic, 0.3 less heteroscedastic, 0.6-moderately heteroscedastic, 0.9-mildly heteroscedastic and 2-severely heteroscedastic. Following Germa et al. (2000), the distribution of the main regression is assumed to be moderately heteroscedastic when variance is proportional to x_1 where x_1 ranged from 11 to 15-mesokurtos) and strongly heteroscedastic where x_2 ranged from 4 to 8-platokurtos.

The error term $U$is generated based on $E(U) = 0$and $E\left(U^2\right) = exp(\delta_0 + \delta_1 x_1 + \delta_2 x_2)^{\delta_i}$ $\delta_i = 0.0; 0.3; 0.6; 0.9$ *and* 2 and δ_0 , δ_1 *and* δ_2 are set at -2, 0.25 and 1 respectively. Thereafter, we incorporated U into the model to generate variable y. The parameters β_0, $\beta_1$and β_2 are set at β_0=10, $\beta_1 = 1$and $\beta_2 = 1$ respectively to generate variable y. The number of replications of our experiment is set at 10,000 with burn-in of 1000 in our MCMC simulation and as well as DMC. For the Bayesian experiment, a Metropolis Hasting Algorithm was developed to simulate our heteroscedastic based models. This was invoked in [R 2.5.2]- a statistical software. The data set class contained the posterior sample for the model parameters. The n.sim option used to specify the number of posterior simulation iteration. The BURN-IN is set at 1000 which specified the draws that were discarded to remove the effect of the initial values. The THINING is set at 5 to ensure the removal of the effect of autocorrelation.

4 Results

Different degrees of heteroscedasticity $\delta = 0$ [homoscedasticity] ; $\delta = 0.3$ [weak heteroscedasticity]; $\delta = 0.6$ [mild heteroscedasticity]; $\delta = 0.9$ [strong heteroscedasticity] and $\delta = 2$ [severe heteroscedasticity] with sample sizes of 25, 50,100 and 200 were considered. Hyper-parameter were arbitrarily chosen for σ^2. Our simulation was based on 10000 iteration for both MCMC and DMC, the level of convergence of the chains were monitored using the method proposed by Gelman et al. (1992) and graphic analysis was carried out using coda package in R package. Multivariate normal and inverse gamma distributions were chosen as priors for parameter estimates and σ^2 respectively. Table 1 reported the mean squares error to compare MLE and BMLE, two type of simulation methods were adopted and we observed that our Bayesian MLE outperformed MLE both in MCMC and DMC. The DMC approach outperformed MCMC approach since the MSE of DMC is always less than MCMC. This is similar to the work of Zellner et al. (2010) in their paper a direct Monte Carlo approach for Bayesian analysis of the seemingly unrelated regression model where they emphasized that DMC performed better than MCMC. Asymptotically, the mean squares error of the parameter estimates in both MCMC and DMC increase algebraically, we inferred that presence of heteroscedasticity in data and model is a serious problem, estimating the parameters without detecting and correcting it will severely affect the inferences no matter the sample. Asymptotically, at each heteroscedastic scale as shown in the charts that appeared at the appendix B that DMC approach outperformed MCMC and MLE estimators.

We examined real dataset, Grunfeld investment data were used due to the presence of heteroscedasticity in it. We confirmed it with Breusch-Pagan heteroscedasticity test and

Table 1: Mean Squares Errors Criterion Measuring Performances of Estimators

Samples	HETERO	MLE	BMLE	DMC
	0	3.546599	-	-
	0.3	1.740209	0.057919	0.059762
25	0.6	0.325895	0.05992	0.059442
	0.9	0.890554	0.059758	0.058875
	2	2.377982	0.059075	0.059263
	0	3.138041	-	-
	0.3	0.299104	0.028517	0.0291
50	0.6	0.168174	0.028932	0.028951
	0.9	0.184701	0.028814	0.028787
	2	1.23099	0.028598	0.028777
	0	2.299636	-	-
	0.3	0.129435	0.014127	0.014266
100	0.6	2.368836	0.014179	0.014108
	0.9	0.186441	0.01421	0.014237
	2	1.338739	0.014224	0.014236
	0	2.608416	-	-
	0.3	0.590969	0.007053	0.007054
200	0.6	0.185827	0.007056	0.007054
	0.9	0.2264	0.007041	0.007045
	2	0.52249	0.007025	0.00705

it was significant with probability value of 6.853e-15. From the analysis we observed that Bayesian Maximum Likelihood Estimator and Direct Monte Carlo outperform Maximum Likelihood estimator as depicted in table 2 above.

5 Conclusion

In this paper we have presented a simple way of modeling and estimating heteroscedastic linear model under two simulation approaches that is MCMC and DMC which were compared with traditional MLE approach. We observed that modeling heteroscedasticity in a full Bayesian improve the precision of the inferences of the estimates. We conclude that DMC approach outperformed MCMC and MLE approaches. Thus MLE performed poorly either asymptotically or in term of scale of heteroscedasticity. Our approach can

Table 2: Application: Grunfeld data

Samples	MLE	BMLE	DMC
220	9052.792	2.710164	2.713274
BP = 65.228	df = 2	p-value = 6.853e-15	

be applied to further studies in the area of simultaneous equation and other econometric methods.

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Appendix A

Table 3: Parameters Estimates of MCMC and DMC Simulation of Heteroscedastic Linear Model [MCMC and MLE]

	Hetero	MLE	Estimates	MCMC Average	Parameter	Estimates	
Samples	λ	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
	0	10	1	1	10.095	2.399	1.394
	0.3	28.51597	0.074831	-0.09732	28.53429	0.36021	0.04091
25	0.6	31.55849	-0.15833	-0.1416	31.56561	-0.06451	-0.09585
	0.9	19.82397	0.566297	0.326572	19.834	0.6867	0.3874
	2	19.72344	0.200021	1.037713	19.7235	0.2014	1.0383
	0	10	1	1	9.804	-1.3658	-0.4182
	0.3	28.98628	0.013437	-0.05911	29.0576	1.0074	0.3497
50	0.6	28.35842	-0.01947	0.11504	28.36641	0.07756	0.17058
	0.9	31.25265	-0.0505	-0.31228	31.2469	-0.1231	-0.3378
	2	27.65699	-0.05131	0.462028	27.65748	-0.04527	0.46406
	0	10	1	1	9.493	-5.185	-2.283
	0.3	29.38964	-0.04888	0.035383	29.3199	-0.9905	-0.2966
100	0.6	26.12378	0.158847	0.117417	26.1476	0.4888	0.1664
	0.9	30.5523	-0.14482	0.083541	30.54	-0.2886	0.007822
	2	29.84105	-0.02426	-0.26731	29.84187	-0.01388	-0.26347
	0	10	1	1	10.019	1.101	1.307
200	0.3	28.97455	-0.02542	0.065491	28.935	-0.579	-0.1571
	0.6	30.01689	-0.04007	-0.0884	30.03097	0.08196	-0.01233
	0.9	30.53187	-0.09533	-0.07727	30.515	-0.3021	-0.1539
	2	31.67139	-0.01991	-0.43032	31.6715	-0.0177	-0.43

Table 4: Parameters Estimates of MCMC and DMC Simulation of Heteroscedastic Linear Model Direct Monte Carlo (DMC)

	Hetero	DMC	Average	parameter Estimates
Samples	λ	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
	0	10.055	1.73	1.357
	0.3	29.6964	1.2507	0.4912
25	0.6	26.8701	0.0358	0.3402
	0.9	29.6735	0.1191	-0.3109
	2	39.8676	-0.3069	-1.278
	0	9.9131	-0.2522	0.6198
	0.3	28.8334	-0.8823	-0.378
50	0.6	26.22345	0.41392	0.06185
	0.9	30.05939	0.09104	-0.31644
	2	30.76646	-0.08749	-0.37222
	0	9.6	-4.287	-1.608
	0.3	28.4039	-0.4365	-0.2175
100	0.6	29.1719	-0.5993	-0.243
	0.9	25.3963	-0.0348	0.1764
	2	33.7611	-0.2698	-0.4069
	0	10.42	6.671	3.506
200	0.3	29.3738	2.4678	0.9956
	0.6	28.3225	-0.7649	-0.3374
	0.9	30.63216	-0.15061	-0.00734
	2	25.2285	0.1035	0.5464

Table 5: Empirical Analysis with Grunfeld dataset

	MLE Estimates		
Samples	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
220	-8.81673	0.109576	0.131478

Table 6: Empirical Analysis with Grunfeld dataset

	MCMC	Average Parameter Estimates	
Samples	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
220	-8.78924	0.109548	0.13127

Table 7: Empirical Analysis with Grunfeld dataset

	DMC Average	parameter	Estimates
Samples	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
220	-8.82089	0.109646	0.131345