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WAVELET COEFFICIENTS CROSS-CORRELATION ANALYSIS OF TIMES SERIES

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Abstract. *The discrete wavelet transform (DWT) is becoming very widely used in the analysis of discrete time stochastic processes. In this paper we explore the maximal overlap discrete wavelet transform (MODWT) which carries out the same filtering steps as the ordinary DWT but does not subsample by 2, and is well defined for any sample size. We address the problem of examining the wavelet auto and cross-correlation structure between wavelet coefficients at different scales of a time series. We construct an estimator of this quantity based on wavelets coefficients. The asymptotic distribution of this estimator is derived for a wide class of stochastic processes.*
A simulation experiment is reported which demonstrates how the cross-correlation spread out over higher scales for linear and nonlinear processes.

Keywords: *Asymptotic distribution, discrete wavelet transform, wavelet autocovariance, wavelet cross-covariance.*

1. Introduction

In this paper we study the wavelet cross-correlation structure between wavelet coefficients at different scales of a discrete time series .The estimation of this quantity is carried out through the use of the maximal-overlap discrete wavelet transform (MODWT) also called the undecimated or shift invariant discrete wavelet transform. The MODWT has been discussed in the wavelet literature see for example [2], [3], [4] and [6].
For the class of discrete compactly supported Daubechies wavelets [3], we denote $\{\tilde{h}_{j,l}, l = 0,1, \dots, L_j - 1\}$ the level j of a wavelet filter of length $L_j = (2^j - 1)(L - 1)$.

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Theses wavelet filters are discussed and given in more details in [4]. Let $\{X_t, t \in Z\}$ be a real valued stationary discrete time stochastic process with autocovariance sequence $s_{X,k}$, the stationary stochastic processes resulting from applying wavelet filters to X_t are given by the level j wavelet coefficients:

$$W_{j,t} = \sum_{l=0}^{L_j-1} \widetilde{h}_{j,l} X_{t-l} \qquad j = 1,2,\ldots,J \tag{1}$$

The wavelet coefficients $W_{j,t}$ are obtained by linear filtering and form a stationary process. The MODWT wavelet coefficient $\widetilde{W}_{j,t}$ based on a finite sample size are given by (2). Each of these coefficients is associated with a particular scale. If we define $\tau_j = 2^{j-1}$, then, $\widetilde{W}_{j,t}$ represents the difference between generalized averages of the finite time series each associated with scale τ_j.
We should note here that the finite sample and subsequently the MODWT algorithm is not yet required here until considering practical estimation.
Let $s_{jk,m} = \mathrm{cov}(W_{j,t}, W_{k,t+m})$ denote the lag m wavelet cross-covariance (ccvs) between $W_{j,t}$ and $W_{k,t+m}$, and $s_{j,m} = s_{jk,m}$ for $j = k$ the lag m wavelet autocovariance (acvs) as defined in [8]. Unlike the acvs $s_{j,m}$, the wavelet ccvs $s_{jk,m}$ was not considered in [8] and no estimation or statistical inferences was derived for this quantity.
By design the filters $\widetilde{h}_{j,l}$ satisfy $\sum_{l=0}^{L_j} \widetilde{h}_{j,l} = 0$ this means that $E(W_{j,t}) = 0$, and therefore $s_{jk,m} = E(W_{j,t} W_{jk,m})$. Our aim is to look at the correlation structure between wavelet coefficients at different scales through the analysis of the wavelet cross-covariance.
The paper is organized as follows. In section 2 we introduce the wavelet auto and cross-covariance estimator based upon the MODWT. The asymptotic distribution of the estimator is derived in section 3. Simulation results are reported in section 4.

2. Wavelet cross-covariance estimator

Assume now we are given a time series that can be regarded as a realization of portion $X_0, X_1, \ldots, X_{N-1}$ of the process X_t. Using a Daubechies wavelet filter with L coefficients we can obtain estimators based upon the MODWT of the series. The MODWT wavelet coefficients based on finite realization are given by

$$\widetilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \widetilde{h}_{j,l} X_{t-l \bmod N} \qquad t = 0,1,\ldots,N-1 \tag{2}$$

obtained as a result of circularly filtering $X_0, X_1, \ldots, X_{N-1}$ with the filter $\tilde{h}_{j,l}$. Where $X_{t\, mod\, N} = X_t$ if $t \geq 0$ and $X_{t\, mod\, N} = X_{N-|t|}$ if $t < 0$. Note that $\tilde{W}_{j,t} = W_{j,t}$ for $t \geq L_j - 1$ and $j = 1, \ldots, J$ where J is the maximum level up to which we run the MODWT . Thus for $j \neq k$ if $Min\{N - L_j, N - L_k\} \geq 0$, we can construct an estimator of the lag m ccvs based upon the MODWT by:

$$\hat{s}_{jk,m} = \begin{cases} \dfrac{1}{M_{jk}} \displaystyle\sum_{t=L_{jk}-1}^{N-m-1} \tilde{W}_{j,t}\, \tilde{W}_{k,t+m} & m = 0,1,\cdots, M_{jk}(N) - 1 \\ 0 & |m| \geq M_{jk}(N) \end{cases} \tag{3}$$

Where $M_{jk} = Max\{M_j, M_k\}$ and $M_j = N - L_j + 1$ and similarly $L_{jk} = Min\{L_j, L_k\}$. We should note here from the choice of M_{jk} and L_{jk} that at most one of the wavelet coefficients $\tilde{W}_{j,t}$ or $\tilde{W}_{k,t}$ will be affected by the circularity operation in (2). An estimator of the lag m ccvs based on all wavelet coefficients including those affected by the circularity can as well be considered but we can easily show that they are asymptotically equivalents and for the sake of simplicity the proof is avoided.

3. Asymptotic Normality

In order to derive the asymptotic distribution of the estimator we assume that the level j wavelet coefficient $W_{j,t}$ given by (1) satisfy the Wald decomposition as in [1]:

$$W_{j,t} = \sum_{i=0}^{\infty} a_{ji} \varepsilon_{j,t-i} \tag{4}$$

where $\{\epsilon_{j,t}\}$ is a sequence of independent identically distributed (i.i.d) random variables for each $j = 1, \ldots, J$ with mean zero, variance σ_j^2, $E(\varepsilon_{j,t}^4) < \infty$ and $Cov(\varepsilon_{j,t}, \epsilon_{k,t'}) = \delta_{tt'} \Delta_{jk}$ where δ is the Kronecker delta, and $\varepsilon_{j,t}$, $\varepsilon_{k,t}$ are correlated at simultaneous times, i.e.
$\Delta_{jk} = Cov(\varepsilon_{j,t}, \varepsilon_{k,t})$ is a real constant. Assume that for each level j the real sequence $\{a_{ji}\}$ is square summable, $\sum_{i=0}^{\infty} a_{ji}^2 < \infty$.
The process $\{X_t\}$ may be linear or nonlinear but its wavelet coefficients are assumed to have a linear representation as in (4). If $\{X_t\}$ is nonlinear, then the linear assumption made for $W_{j,t}$ remains valid for so many processes, because of the filtering operation. Now consider the statistic:

$$T_{N,m}^{(jk)} = \sum_{t=1}^{N} W_{j,t} W_{k,t+m} \qquad m = \ldots, -1, 0, 1, 2 \ldots \tag{5}$$

Proposition
Let $\{X_t\}$ be a real valued discrete time stochastic process, and $W_{j,t}$ the mean-zero linear process as defined in (4). Then the statistic:

$$N^{\frac{-1}{2}}\left(T_{N,m}^{(jk)} - E\left(T_{N,m}^{(jk)}\right)\right) \tag{6}$$

is asymptotically normal with mean zero and variance:

$$\sigma_{jk}^2(m) = \sum_{l\in Z} s_{j,l}s_{k,l} + \frac{\Delta_{jk}^2}{\sigma_j^2\sigma_k^2}\sum_{l\in Z} s_{jk,l}s_{jk,2m-l} + \left(\frac{E\left(\varepsilon_{j,0}^2\varepsilon_{k,0}^2\right)}{\Delta_{jk}^2} - \frac{\sigma_j^2\sigma_k^2}{\Delta_{jk}^2} - 2\right)s_{jk,m}^2 \tag{7}$$

The proof follows readily from the theorem and lemma given in page 42 of [9]. Let $Z_{t,m}^{(jk)} = W_{j,t}\, W_{k,t+m} - E(W_{j,t}\, W_{k,t+m})$ and $S_{Z_m}^{(jk)}(f)$ the power spectral density function (sdf) of the process $\{Z_{t,m}^{(jk)}\}_t$, then we can easily show that $\sigma_{jk}^2(m) = S_{Z_m}^{(jk)}(0)$. As a consequence, the estimator of the lag-m cross-covariance defined in (3) is asymptotically Gaussian distributed with mean zero and

$$\lim_{N\to\infty} \frac{M_{jk}^2(N)}{M_{jk}(N,m)} var\left(\hat{s}_{jk,m}\right) = S_{Z_m^{(jk)}}(0) \tag{8}$$

By letting the levels $j = k$ we are considering the case of the lag m sample autocovariance $\hat{s}_{jk,m} = \hat{s}_{j,m}$ within the same level i.e. of $W_{j,t}$ which is discussed in [9].
If $\varepsilon_{j,t}$ for $j = 1, \dots, J$ are Gaussian innovations, it well known that:

$$E\left(\epsilon_{j,0}^2\ \epsilon_{k,0}^2\right) = \sigma_j^2\sigma_k^2 + 2\Delta_{jk}^2$$

and the asymptotic variance (7) of the statistic (6) gives:

$$\sigma_{jk}^2(m) = \sum_{l\in Z} s_{j,l}s_{k,l} + \frac{\Delta_{jk}^2}{\sigma_j^2\sigma_k^2}\sum_{l\in Z} s_{jk,l}s_{jk,2m-l} \tag{9}$$

4. Simulations

To illustrate the asymptotic properties of the estimator, we look at three different models, and assume that their corresponding wavelet coefficients $W_{j,t}$ satisfy the requirements of the previous proposition. We simulate a Gaussian model, an AR(1) model, $X_t - 0.7X_{t-1} = \epsilon_t$, where ϵ_t is a Gaussian uncorrelated sequence with mean zero and variance $\sigma^2 = 1$, and a non-Gaussian nonlinear model and estimate the large sample standard deviation $\sqrt{S_{Z_m^{(jk)}}(0)\frac{M_{jk}(N,m)}{M_{jk}^2(N)}}$ as given by (8). The method for simulating non-Gaussian nonlinear processes is developed in [5], [11] and illustrated as in [10]. Table 1 gives the three models and their autocovariance sequences $s_{X,k}$.

Table 1. Simulated models with their acv sequences.

Model	Autocovariance
1. Gaussian: X_t	$s_{X,k} = \cos(0.1)\, e^{-0.2\,\lvert k\rvert}$
2. $AR(1)$: $X_t - \varphi\, X_{t-1} = \epsilon_t$	$s_{X,k} = \varphi^k s_{X,0}, \quad \varphi = 0.7 \; and \; s_{X,0} = \frac{1}{1-\varphi^2}$
3. Nonlinear: $Y_t = X_t^2 - 1$	$s_{Y,k} = 2s_{X,k}^2 \;\; and \; s_{X,k} = \cos\,(0.2k)e^{-0.1\lvert k\rvert}$

For each model, a series $\{X_t, t = 0, \dots, N-1\}$ of length $N = 1000$ was generated. Simulations were carried out for levels $j = 1, \dots, J$ with $J = 5$ using filter coefficient of the least-asymmetric Daubechies wavelet filter of length $L = 8$.

(i) For each level $j, k \in \{1, \dots, J\}$ sets of $M_j(N) = N - L_j + 1$ and $M_k(N) = N - L_k + 1$ coefficients from $\widetilde{W}_{j,t} = W_{j,t}, t = L_j, \dots, N-1$ and $\widetilde{W}_{k,t} = W_{k,t}, t = L_k, \dots, N-1$ where extracted and used to compute the MODWT auto- and cross-covariances: $\hat{s}_{j,m}$ and $\hat{s}_{jk,m}$ at lags $m = 0,1,2,4,8,15,25$

(ii) For $j, k \in \{1, \dots, J\}$, we extracted the sequences $\widetilde{W}_{j,t}\,\widetilde{W}_{k,t+m}$, $t = L_{jk}, \dots, N-1$ from which the expression (3) and hence the asymptotic variance $\sigma_{jk}^2(m)$ of the proposition was estimated as $\hat{S}_{Z_m^{(jk)}}(0)$ and $\hat{S}_{Z_m^{(j)}}(0)$ if $j = k$ via the multitaper spectrum analysis method using $K = 5$ tapers and design bandwidth $2W = \frac{7}{M_{jk}(N,m)}$, where $M_{jk}(N,m) = N - L_{jk} - m + 1$. (These choices conform the standard practice see [7], Chapter 7.)

(iii) steps (i) and (ii) were repeated 100 times and from these repetitions were calculated the standard error of $\hat{s}_{jk,m}$ for $j = 1,2,3,4$ and $k = 1,2,3,4,5,6$ for lags $m = 0,1,2,4,8,15,25$. Also computed was the average of the estimated standard deviations, derived from (3) via the spectrum estimate at zero frequency, for the same values of levels j, k and lags m.

True values of wavelet cross-covariances were calculated from the quadratic form

$$s_{jk,m} = \sum_{i,l}^{L_{jk}-1} \widetilde{h}_{j,l}\, \widetilde{h}_{k,i+m}\, s_{X,m+i-l} \qquad (10)$$

In order to assess the performance of the estimator we give a plot of the estimator $\hat{s}_{j,m}$ for the three models. For brevity we give the plot only for few contributing scales. Figure 1 displays the true values along with the estimate of wavelet auto- and cross-covariance at lags

$m = 0,1,2,4,8,15,25$ for levels $j = 1,2,3$ and $k = 1,2,4,5$. The crosses in all plots are the true values $s_{jk,m}$ given by (10) for each model, the dashed lines are the corresponding estimates. In most plots there is a near perfect agreement between the true and the estimated value. However some curvature has appeared at some lags in the cross levels $(j,k) = (3,5)$ in figure 1. But we should note here that this occur because we are comparing correlation between wavelet coefficients not only at different scales, but at scale getting away from each other.

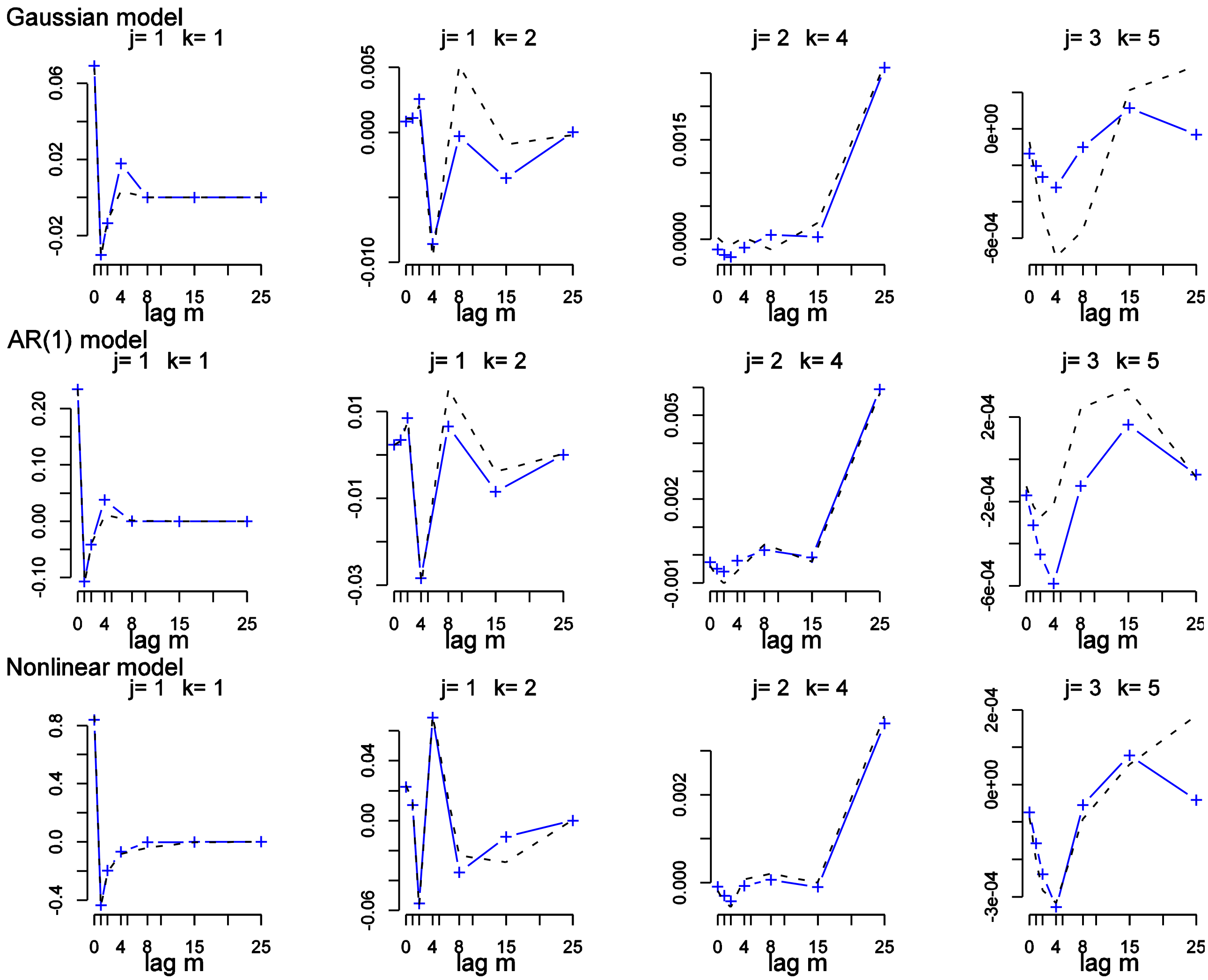

Figure 1. True (crosses) and estimated (dotted line) wavelet autocovariance and cross-covariance at lags $m = 0,1,2,4,8,15,25$ for different levels j and k, for the Gaussian, AR(1) and Nonlinear time series.

Comparing correlation strength between levels we can see that both wevelet autocovariances and cross-covariances show a small correlation over a number of lags that start to vanish as we move to higher scales .

The average of the estimated asymptotic standard deviations computed in simulations are plotted for each model and given in Figures 2 (dotted lines). Also shown in each plot (crosses) are the standard errors of $\hat{s}_{j,m}$ and $\hat{s}_{jk,m}$ over the 100 simulations.

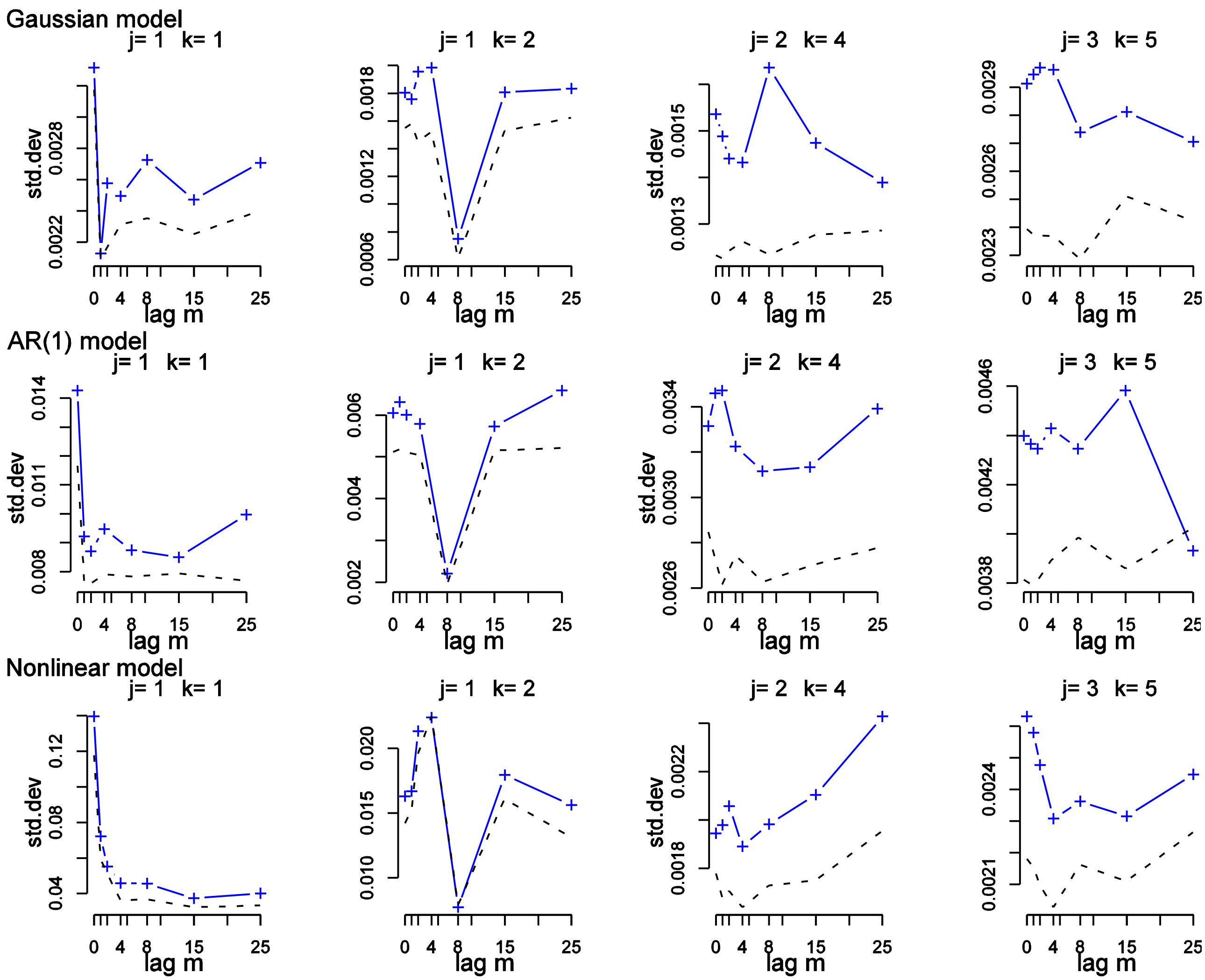

Figure 2. The average estimated standard deviations (dotted line), computed from the spectrum estimate at frequency zero and the standard errors from the 100 simulations (crosses) for $\hat{s}_{j,m}$ and $\hat{s}_{jk,m}$ at lags $m = 0,1,2,4,8,15,25$.

In most plots the estimated asymptotic standard deviations agree quite well with the simulated standard errors. In some cases for instance for $(j,k) = (2,4)$ and $(j,k) = (3,5)$ at lags $m = 0,1,2,8$, in the Gaussian and AR(1) model, these values disagree to some extend. This is due to the spectrum estimate at zero frequency. However, the estimated asymptotic standard deviations confirm the practical utility of expression (8).

5. Conclusion

The simulation results illustrate and support the theory. First there is very good agreement between $s_{jk,m}$ and the average of $\hat{s}_{jk,m}$ over 100 simulations for all models. Second, The multitaper method provide a reasonable estimation of $S_{Z_m^{(jk)}}(0)$. The over all performances of the estimator are good enough except in relatively few cases. We can see as well that the most important feature as illustrated in all three models is that a decorrelation is occurring within the same scales, and getting even smaller in between different scales.

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