Ordinal utility of moments: foundations and financial behavior

Riccardo Cesari
Università di Bologna, Dip. MatemateS, viale Filopanti, 5
riccardo.cesari@unibo.it

Carlo D'Adda
Università di Bologna, Dip. Scienze economiche, strada Maggiore, 45
carlo.dadda@unibo.it

Abstract: *After Tobin (1958), a considerable effort has been devoted to connecting the expected utility approach to a utility function directly expressed in terms of moments. We follow the alternative route of providing, for the first time, the theoretical, autonomous foundation of an ordinal utility function of moments, representing rational choices under uncertainty, free of any 'independence axiom' and compatible with all the behavioral "paradoxes" documented in the economic literature.*
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1. Introduction

In this paper we develop for the first time - to our knowledge – the foundation of an *ordinal* utility of moments (section 2) as a rational and autonomous criterion of choice under uncertainty, showing that it explains all the best known behavioral paradoxes which are still embarrassing the expected utility theory (section 3). This ordinal approach is strongly reminiscent of standard microeconomic theory and it could be used to reset and generalize both assets demand and asset pricing models.

2. Foundations of an ordinal utility of moments for decisions under uncertainty

It is well known that the theory of choice under uncertainty assumes that preferences are defined over the set of probability distribution functions (e.g. Savage, 1954 ch. 2 and DeGroot, 1970 ch. 7).
In particular, let $(\Omega,\Im,\wp)$ be a standard probability space, Ω being the set of elementary events (states of the world), $\Im$ the set of subsets of Ω (events), $\wp$ a (subjective) probability measure of the events. Given the set A of all possible actions or decisions, all couples (ω,a) with $\omega\in\Omega$ and $a\in A$, are mapped onto a real vector of monetary consequences $c\in R^n$, the Euclidean space of n-dimensional real vectors, so that $X(\omega,a)=c$ or $X_a(\omega)=c$ is a random variable and $F_a\in\mathcal{F}$ is its probability distribution function. Clearly, the preferences over acts in A are, equivalently, preferences over the set of random variables X_a and preferences over the set $\mathcal{F}$ of distribution functions. Let us confine ourselves, for ease of exposition, to the case of univariate distributions (n=1) and assume that the essential information concerning any distribution F is contained in the m-dimensional vector of moments $\mathbf{M}\equiv(\mu, \mu^{(2)}, \mu^{(3)}, \ldots, \mu^{(m)})$ where μ is the mean, and $\mu^{(s)}$ is the s-order central moment in original units[1]:

Definition of s-order modified central moment:
$$\mu^{(s)} \equiv \begin{cases} 1 & s=0 \\ \mu & s=1 \\ \left(\int (x-\mu)^s \, dF\right)^{\frac{1}{s}} & s\geq 2 \end{cases} \qquad (2.1)$$

[1] Note that, instead of central moments, noncentral moments could, equivalently, be used. Moreover, scale, location and dispersion parameters can be considered in the case of distributions (e.g. stable) for which moments do not exist.

Note that $(\mu^{(s)})^s$ is the usual central moment of order s≥2.

Let $Q \subseteq R^m$ be a rectangular subset of R^m (the Cartesian product of m real intervals), whose elements are the m-dimensional vectors of moments, $\mathbf{M} \in Q$. Following to Fishburn (1970) we show the existence of an ordinal utility of moments and define its properties.

Assumption of preference order: Let $\succ$ be a preference order i.e. a binary relation defined by a subset $\Re$ of the Cartesian product QxQ, whose elements are the ordered pairs of vectors $(\mathbf{M}_a,\mathbf{M}_b)$.

We write $\mathbf{M}_a \succ \mathbf{M}_b$ instead of $(\mathbf{M}_a,\mathbf{M}_b) \in \Re$ and we say that "$\mathbf{M}_a$ is preferred to $\mathbf{M}_b$", corresponding to "F_a is preferred to F_b".

Clearly, or $\mathbf{M}_a \succ \mathbf{M}_b$ or $\mathbf{M}_a \nsucc \mathbf{M}_b$ and both cannot hold: in fact, or $(\mathbf{M}_a,\mathbf{M}_b) \in \Re$ or $(\mathbf{M}_a,\mathbf{M}_b) \notin \Re$.

I. Axiom of asymmetric preferences: We assume that $\succ$ is asymmetric i.e. that:

if $\mathbf{M}_a \succ \mathbf{M}_b$ then $\mathbf{M}_b \nsucc \mathbf{M}_a$ (2.2)

Note that if $\mathbf{M}_b \nsucc \mathbf{M}_a$ then two alternative cases are possible: either $\mathbf{M}_a \succ \mathbf{M}_b$ or $\mathbf{M}_a \nsucc \mathbf{M}_b$.

In the latter case we say that "$\mathbf{M}_a$ and $\mathbf{M}_b$ are equivalent" and we write $\mathbf{M}_a \sim \mathbf{M}_b$.

Definition of equivalence: $\mathbf{M}_a \sim \mathbf{M}_b$ iff $\mathbf{M}_a \nsucc \mathbf{M}_b$ and $\mathbf{M}_b \nsucc \mathbf{M}_a$

Theorem of complete preferences: Given $\mathbf{M}_a$, $\mathbf{M}_b \in Q$ then one and only one case holds:

$\mathbf{M}_a \succ \mathbf{M}_b$ or $\mathbf{M}_b \succ \mathbf{M}_a$ or $\mathbf{M}_a \sim \mathbf{M}_b$.

Proof: It is easy to show that any two cases are a contradiction. ■

II. Axiom of transitive preferences: We assume that $\succ$ is transitive:

if $\mathbf{M}_a \succ \mathbf{M}_b$ and $\mathbf{M}_b \succ \mathbf{M}_c$ then $\mathbf{M}_a \succ \mathbf{M}_c$ (2.3)

Definition of weak order: The preference order $\succ$ is a weak order if it is asymmetric and transitive.

Definition of negatively transitive preferences: if $\mathbf{M}_a \nsucc \mathbf{M}_b$ and $\mathbf{M}_b \nsucc \mathbf{M}_c$ then $\mathbf{M}_a \nsucc \mathbf{M}_c$.

Lemma: $\succ$ is negatively transitive if and only if, for every $\mathbf{M}_a$, $\mathbf{M}_b$, $\mathbf{M}_c \in Q$, $\mathbf{M}_a \succ \mathbf{M}_b$ implies $\mathbf{M}_a \succ \mathbf{M}_c$ or $\mathbf{M}_c \succ \mathbf{M}_b$.

Proof: Under negative transitivity, if $\mathbf{M}_a \succ \mathbf{M}_b$ but $\mathbf{M}_a \nsucc \mathbf{M}_c$ and $\mathbf{M}_c \nsucc \mathbf{M}_b$ then by negative transitivity $\mathbf{M}_a \nsucc \mathbf{M}_b$ against the assumption.

Viceversa, if $\mathbf{M}_a \succ \mathbf{M}_b$ implies $\mathbf{M}_a \succ \mathbf{M}_c$ or $\mathbf{M}_c \succ \mathbf{M}_b$ and negative transitivity is false we have $\mathbf{M}_a \nsucc \mathbf{M}_b$, $\mathbf{M}_b \nsucc \mathbf{M}_c$ but $\mathbf{M}_a \succ \mathbf{M}_c$. Therefore $\mathbf{M}_a \succ \mathbf{M}_b$ or $\mathbf{M}_b \succ \mathbf{M}_c$ against the assumption. ■

Theorem of negatively transitive preferences: Asymmetric and transitive preferences are equivalent to asymmetric and negatively transitive. Moreover, the equivalence $\sim$ is reflexive, symmetric and transitive and $\succ$ on $Q|\sim$ (the set of equivalence classes of Q under $\sim$) is a strict order in the sense that for every equivalent class $\mathbf{M}_A$, $\mathbf{M}_B \in Q|\sim$ one and only one case holds: $\mathbf{M}_A \succ \mathbf{M}_B$ or $\mathbf{M}_B \succ \mathbf{M}_A$.

Proof: Under transitivity if $\mathbf{M}_a \succ \mathbf{M}_b$ and $\mathbf{M}_b \succ \mathbf{M}_c$ then $\mathbf{M}_a \succ \mathbf{M}_c$; therefore, by asymmetry, if $\mathbf{M}_b \nsucc \mathbf{M}_a$ and $\mathbf{M}_c \nsucc \mathbf{M}_b$ then $\mathbf{M}_c \nsucc \mathbf{M}_a$ which is negative transitivity.

Viceversa, under negative transitivity, if $\mathbf{M}_a \succ \mathbf{M}_b$ and $\mathbf{M}_b \succ \mathbf{M}_c$ then, from previous Lemma, ($\mathbf{M}_a \succ \mathbf{M}_c$ or $\mathbf{M}_c \succ \mathbf{M}_b$) and ($\mathbf{M}_b \succ \mathbf{M}_a$ or $\mathbf{M}_a \succ \mathbf{M}_c$). But $\mathbf{M}_c \succ \mathbf{M}_b$ and $\mathbf{M}_b \succ \mathbf{M}_a$ are false by asymmetry. Therefore $\mathbf{M}_a \succ \mathbf{M}_c$ which means transitivity.

The equivalence is clearly reflexive and symmetric. Suppose it is not transitive: $\mathbf{M}_a \sim \mathbf{M}_b$ and $\mathbf{M}_b \sim \mathbf{M}_c$ but $\mathbf{M}_a \sim \mathbf{M}_c$ is false. Then, by definition, either $\mathbf{M}_a \succ \mathbf{M}_c$ or $\mathbf{M}_c \succ \mathbf{M}_a$. From the Lemma, in the first case, $\mathbf{M}_a \succ \mathbf{M}_b$ or $\mathbf{M}_b \succ \mathbf{M}_c$; in the second case $\mathbf{M}_c \succ \mathbf{M}_b$ or $\mathbf{M}_b \succ \mathbf{M}_a$, in contradiction with the hypothesis.

For the proof of strict order of $\succ$ on $Q|\sim$ see Fishburn (1970, p. 13). ■

III. Axiom of continuity: There is a countable subset $D \subseteq Q|\sim$ that is $\succ$-dense in $Q|\sim$ i.e. for every $\mathbf{M}_A$, $\mathbf{M}_C \in Q|\sim \backslash D$, $\mathbf{M}_A \succ \mathbf{M}_C$ there is $\mathbf{M}_B \in D$ such that:

$\mathbf{M}_A \succ \mathbf{M}_B$ and $\mathbf{M}_B \succ \mathbf{M}_C$ (2.4)

Note that the subset of rational numbers is >-dense and <-dense in the set of real numbers.

Theorem of ordinal utility on moments: Under Axioms I, II, III there is a real function

$H: Q \longrightarrow R$ which represents the preferences $\succ$, i.e. such that for every $\mathbf{M}_a$, $\mathbf{M}_b \in Q$

$\mathbf{M}_a \succ \mathbf{M}_b$ if and only if $H(\mathbf{M}_a) > H(\mathbf{M}_b)$

The function H is unique up to any order-preserving transformation Ψ:
$H(\mathbf{M}_a) > H(\mathbf{M}_b)$ if and only if $\Psi(H(\mathbf{M}_a)) > \Psi(H(\mathbf{M}_b))$.
Proof: Fishburn (1970, p.27). ■

3. Indifferent pricing and the (dis)solution of paradoxes

The utility of moments is perfectly compatible with the empirical observations in all well known behavioral paradoxes, described in terms of games and lotteries.
In order to show this compatibility we use the so called utility indifferent pricing (Henderson and Hosbon, 2004) which can be applied in all cases of *personal valuation* of non traded assets and incomplete markets.
For the sake of simplicity, let us consider the two-moment ordinal utility.
Let W be current wealth of the decision maker and $\tilde{G}$ be the random variable representing the game, with mean M_G and vol Σ_G. Future wealth, in case of a decision to gamble, is given by:

$$\tilde{W} = \frac{W - P_G}{P_0} + \tilde{G} \qquad (3.3)$$

where $W\text{-}P_G$ is wealth left after payment for the game, invested at the riskless rate $r = \frac{1}{P_0} - 1$ (often set to zero) and the personal indifferent price P_G is defined as the price at which the agent is indifferent between paying the price and entering the game and paying nothing and avoiding the game:

$$H\left(\frac{W - P_G}{P_0} + M_G, \Sigma_G\right) = H\left(\frac{W}{P_0}, 0\right) \qquad (3.4)$$

In the l.h.s., using a Taylor series approximation for small risks (Pratt, 1964) we have:

$$H\left(\frac{W - P_G}{P_0} + M_G, \Sigma_G\right) = H\left(\frac{W}{P_0}, 0\right) + \frac{\partial H}{\partial M}\left(M_G - \frac{P_G}{P_0}\right) + \frac{\partial H}{\partial \Sigma}\Sigma_G \qquad (3.5)$$

so that, imposing (4.4) and simplifying, we obtain:

$$P_G = M_G P_0 + \left(\frac{\partial H / \partial \Sigma}{\partial H / \partial M} P_0\right)\Sigma_G \qquad (3.6)$$

where the term in brackets is the (negative) personal price of the vol P_σ . Therefore, the reservation price of the game is obtained as moment quantities, M, Σ, times subjective moment prices, P_0 , P_σ.
Moreover, equation (3.6) can be easily generalized to higher moments (skewness Γ and kurtosis Ψ):

$$P_G = M_G P_0 + \Sigma_G P_\sigma + \Gamma_G P_\varsigma + \Psi_G P_\kappa \qquad (3.7)$$

and it will be used in the following to show that the behavior of a utility of moment maximizer is perfectly compatible with all proposed paradoxes, from St. Petersburg (1713) to Allais (1953, 1979), Ellsberg (1961) and Kahneman and Tversky (1979, 1981).

3.4 The Kahneman and Tversky (1979) paradox. In a famous experiment, a systematic violation of the independence axiom was documented: 80% of 95 respondents preferred A to B where:

$$A = \{3000 \quad 100\% \qquad\qquad B = \begin{cases} 0 & 20\% \\ 4000 & 80\% \end{cases}$$

65% preferred B' to A' where:

$$A' = \begin{cases} 0 & 75\% \\ 3000 & 25\% \end{cases} \qquad\qquad B' = \begin{cases} 0 & 80\% \\ 4000 & 20\% \end{cases}$$

and more than 50% of respondents violated the independent axiom, given that, if Q pays 0 for sure, then[2]:

$$A' = \begin{cases} Q & 75\% \\ A & 25\% \end{cases} \qquad B'' = \begin{cases} Q & & 75\% \\ B = \begin{cases} 0 & 20\% \\ 4000 & 80\% \end{cases} & & 25\% \end{cases}$$

and B'' is considered equal to B' in terms of outcomes and probabilities.
The point is that, in terms of valuation, B' and B'' are not the same asset, and B'' is equivalent to:

$$B'' = \begin{cases} 0 & 75\% \\ P(B) & 25\% \end{cases} \quad \neq \quad B' = \begin{cases} 0 & 80\% \\ 4000 & 20\% \end{cases}$$

Using the first four moments:

	A'	B	B'
mean, μ	750	3200	800
standard deviation, σ	1299.04	1600	1600
skewness, ς	1362.84	-1831.54	1831.54
kurtosis, κ	1605.52	2148.28	2148.28

and assuming the following prices of the four moments:
$P_0=1$, $P_\sigma=-0.2$, $P_\varsigma=0.1$, $P_\kappa=-0.001$ we obtain the prices of the lotteries: P(A)=3000 > P(B)=2694.70 and P(A')=624.87 < P(B')=661.01, in accordance with the experimental results. Note also that P(B'')=561.28<P(A')<P(B').

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[2] Note that treating lotteries as assets implies that linear combinations such as 0.75Q+0.25A are meaningful and P(0.75Q+0.25A)=0.25P(A)≠P(A').

