Risk processes with delayed claims for heavy tailed distributions

Gabriele Stabile
Dipartimento di Matematica per le Appl. Econ. Finanz. Assic.
Università di Roma "Sapienza"

Giovanni Luca Torrisi
Istituto per le Applicazioni del Calcolo "Mauro Picone" (IAC), Consiglio Nazionale delle Ricerche (CNR)

Abstract

We study the ruin problem for a non-life insurance company, whose risk process has the following features: claims are not immediately settled, heavy tailed claims distributions. We prove a large deviation principle for the total claim amount process, and then we provide an asymptotic estimates for the logarithm of the infinite horizon ruin probabilities.

Keywords: Ruin probabilities, Shot noise process, Semiexponential distributions, Large deviations.

1 Introduction

This paper studies risk processes with two main features: claims are not immediately settled (delayed claims), heavy tailed claim sizes. The I.B.N.R. (*Incurred but not reported*) claims are an example of delayed claims. These are claims not yet known to the insurer, but for which a liability is believed to exist at the reserving date. Building reserves for these claims is an important issue in the financial statement of an insurance company. Heavy-tailed probability distributions have a key role in non-life insurance since they describe phenomena where there is a significant probability that a single claim have a great impact on the ruin of the company.

Stochastic models for I.B.N.R. claims, which require payments over several years until they are finally settled, have been proposed by Waters and Papatriandafylou (1985) and. more recently, Yuen *et al.* (2005). In Klüppelberg and Mikosch (1995), Brémaud (2000) Klüppelberg *et al.* (2003) the shot-noise process is proposed as a natural model for delay in claim settlement and in this paper we adopt this choice. More precisely, we describe the total claim amount by the Poisson shot noise

$$S(t) = \sum_{n=1}^{N_t} H(t - T_n, Z_n) \qquad (1)$$

where $(N_t)_{t\geq 0}$ is a Poisson process on $\mathbb{R}^+$ with intensity λ and epochs of jumps $(T_k)_{k\geq 1}$, $(Z_k)_{k\geq 1}$ is a sequence of i.i.d. E-valued random variables independent of the Poisson process, and $H : \mathbb{R}^+ \times E \to [0, \infty[$ is a measurable function which is continuous and increasing with respect to the first coordinate. The epoch T_n represents the instant when the nth claim occurs and the process $(H(t, Z_n))$ describes its pay-off procedure.

The aim of this paper is to study the asymptotic behavior of ruin probabilities $\psi(u) := \mathrm{P}\{\inf_{t\geq 0} R(t) < 0\}$, for the risk process

$$R(t) = u + ct - S_t \tag{2}$$

where $u > 0$ is the initial capital, $c > 0$ is the constant premium rate.

Under the light tail assumption, this problem was addresses in Brémaud (2000), where large deviation theory was used to give a Cramér-Lundberg-type estimates of the infinite horizon ruin probability. In the case of heavy-tailed claims, the Cramér-Lundberg condition is not valid, and then traditional large deviation theory cannot be used. Ganesh and Torrisi (2006) give asymptotic results for $\Psi(u)$, as $u \to \infty$ for a class of subexponential claims. They assume a specific multiplicative shot, *i.e.* $H(t - T_k, Z_k)) = F(t - T_k)Z_k$ where $F(\cdot)$ is a distribution function with $F(t) = 0$ for $t < 0$, and they model claims arrival with a general renewal process.

In this paper we provide asymptotic estimates for $log\Psi(u)$, as $u \to \infty$, for a class of semiexponential claims, including the heavy-tailed Weibull distribution. We consider a general family of shot shape including the multiplicative shot considered in Ganesh and Torrisi (2006). On the other hand, our model considers only Poisson arrivals.
First, we present a Large Deviation Principle (LDP) for the Poisson shot noise process in case of semiexponential claims distributions. Then, we provide an asymptotic estimates for $log\Psi(u)$, as $u \to \infty$. Finally, an example is presented.

2 Asymptotic results for ruin probabilities

In this section we want to provide an asymptotic estimates of the ruin probability for the risk process (2). At this aim, we first present a LDP for the Poisson shot noise process (1). From now on, we denote with $x_0 = \lambda \mathrm{E}[H(\infty, Z)]$.

Proposition 2.1 *Let a and L be slowly varying functions, and take $r \in (0, 1)$.*
(i) *If exists $k \in (0, 1)$ such that*

$$\liminf_{t\to\infty} \frac{1}{t^r L(t)} \log \mathrm{P}\big(H(tk, Z_1) \geq tb\big) \geq -b^r \qquad \forall\ b > 0 \tag{3}$$

then

$$\liminf_{t\to\infty} \frac{1}{t^r L(t)} \log \mathrm{P}\Big(\frac{S(t)}{t} \geq x\Big) \geq -(x - x_0)^r, \quad x > x_0$$

(ii) *If*

$$\mathrm{P}\big(H(\infty, Z) \geq x\big) \leq a(x) \exp\big(- L(x)x^r\big) \tag{4}$$

then

$$\limsup_{t\to\infty} \frac{1}{t^r L(t)} \log \mathrm{P}\Big(\frac{S(t)}{t} \geq x\Big) \leq -(x - x_0)^r, \quad x > x_0$$

(iii) *If (3) and (4) hold then $\left(\frac{S(t)}{t}\right)_{t>0}$ obeys a LDP on $\mathbb{R}$ with speed $t^r L(t)$ and good rate function*

$$I(x) = \begin{cases} (x - x_0)^r & \text{if } x \geq x_0 \\ +\infty & \text{if } x < x_0. \end{cases}$$

The proof, which is here omitted, makes use of techniques used in Gantert (1996), in which a LDP is proved for a mixing sequence of random variables. We notice that, by applying the contraction principle, we have that $(\frac{R(t)+}{t})$ satisfies a LDP on $\mathbb{R}$ with speed $t^r L(t)$ and good rate function

$$\tilde{J}(y) = \inf\left\{I(x) : F(x) = y\right\} = \begin{cases} (y + c - x_0)^r & \text{if } y \geq x_0 - c \\ +\infty & \text{if } x < x_0 - c. \end{cases}$$

where $F(x) = x - c$ is a continuous function.
Let us define the infinite horizon ruin probability

$$\Psi(u) = \mathrm{P}(\tau < \infty)$$

where

$$\tau = \inf\left\{t \geq 0 : R(t) \geq u\right\}$$

and $\inf \emptyset = \infty$ by convention.
We want provide an asymptotic estimate for $\log \Psi(u)$. At this aim, we use Proposition 2.1 in Duffy *et al.* (2003). We assume that the constant premium rate c satisfies the net profit condition (NPC) $c > x_0$. From now on, let B_0 denote the distribution function of the integrated tail of $H(\infty, Z)$ and $\bar{B}_0(u)$ denote the tail of B_0.

Proposition 2.2 *Suppose that the hypothesis of Proposition 2.1 are satisfied. Moreover, let us suppose that:*
(i) B_0 *is subexponential;*
(ii) $\log \bar{B}_0(u) \sim -u^r L(u)$
Then

$$\lim_{u \to \infty} \frac{1}{u^r L(u)} \log \Psi(u) = -1$$

We omit the proof for the sake of brevity.
Notice that in the Compound Poisson case, if the tail of Z is $\mathrm{P}\big(Z > x\big) = \exp(-x^2), r \in (0,1)$, we know that $\Psi^0(u) \sim C u^{1-r} \exp(-u^2)$, where C is a constant and Ψ^0 is the ruin probability in the Compound Poisson case (see Asmussen and Kluppelberg (1996)). Equivalently, we may write $\frac{1}{u^r} \log \Psi^0(u) \longrightarrow -1$, as $u \to \infty$. Thus, Proposition 2.2 gives an insensitivity property because the asymptotic behavior of $\Psi(u)$ depends only on the distribution of $H(\infty, Z)$, not on the shape shot.

Example

Let us assume either $H(t,z) = F(t)z$ or $\sup_{z \geq 0} |H(t,z) - H(\infty, z)| \longrightarrow 0$, as $u \to \infty$. The former type represents multiplicative shot shapes, whereas the latter type characterizes a kind of uniform shot shapes. Moreover, let us consider $\mathrm{P}\big(H(\infty, z) \geq x\big) = \exp(-x^2 L(x)), r \in (0,1)$ with $L(x) \longrightarrow \gamma > 0$, as $x \longrightarrow \infty$, and $L(\cdot)$ differentiable such that $xL'(x) \longrightarrow 0$. Finally, assume the NPC holds. Then $log \Psi(u) \sim -u^r L(u)$.
Notice that, $H(t,z) = (Z - f(t))^+$, with $f(t)$ continuous function on $(0, \infty)$ such that it converges to a constant at infinity, belongs to the class of the uniform shot shapes. It is a shot shape relevant for the reinsurance models, because it describes the *per risk excess of loss* reinsurance form.

References

Asmussen, S. and Klüppelberg, C. (1995), Large deviation results for subexponential tails, with applications to insurance risk *Stoch. Process. Appl.* **64**, 103–125.

Brémaud, P. (2000), An insensitivity property of Lundberg's estimate for delayed claims, *J. Appl. Probab.* **37**, 914–917.

Duffy, K., Lewis, T.L. and Sullivan, W.G. (2003), Logarithmic asymptotics for the supremum of a stochastic process, *The Annals of Applied Probability* **13**, No. 2 430–445.

Ganesh, A. and Torrisi, G.L. (2006), A class of risk processes with delayed claims:ruin probability estimates under heavy-tail conditions, *J. Appl. Probab.* **43**, 916–926.

Gantert, N. (1996), Large deviations for heavy-tailed mixing sequence, Preprint.

Klüppelberg, C. and Mikosch, T. (1995), Explosive Poisson shot noise with application to risk reserves, *Bernoulli* **1**, 125–147.

Klüppelberg, C., Mikosch, T. and Schärf, A. (2003), Regularly varying in the mean and stable limits for Poisson shot noise, *Bernoulli* **9**, 467–496.

Waters, H.R. and Papatriandafylou, A. (1985), Ruin probabilities allowing for delay in claims settlement, *Insurance Math. Econom.* **4**, 113–122.

Yuen, K.C., Guo, J. and Ng, K.W. (2005), On ultimate ruin in a delayed-claims risk model, *J. Appl. Probab.* **42**, 163–174.

