Evolution of Coalition Structures under Uncertainty

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Abstract: *In this paper we introduce and investigate some evolutionary games arising from the γ and the δ static models of coalition formation defined in Hart and Kurz (1983). In particular, we assume that players determine at every instant their strategies and we study how the coalition structure might evolve accordingly. Finally, we focus on minimal change revisions of coalition structure probabilistic beliefs and provide existence results.*

Keywords: coalition formation; coherent beliefs; differential inclusions; viability theory; minimal change belief revision.

1. Introduction

In Hart and Kurz (1983), stability of coalition structures has been analyzed via concepts of equilibrium in associated strategic form games. The key feature for this approach is that the strategy set of each player *i* is the set of all subgroups of players containing *i* and his choice represents the coalition he wishes to join. However, given a strategy profile (i.e. a coalition for each player), the coalition structure formed is not unequivocally determined. So, different rules of coalition structure formation can be considered, namely, functions associating to every strategy profile a coalition structure. In Hart and Kurz (1983) the following rules are proposed: the model γ and the model δ. In this paper we look at the evolutionary games arising from the γ and the δ models in which players determine at every instant their strategies and, in particular, we study how the coalition structure evolve according to the strategic choices. For this purpose we consider mixed strategies and, firstly, we notice that natural generalizations of the γ and the δ models to this case lead to multiplicity of beliefs (on the set of coalition structures) coherent with the probability assignments given by the strategy profile. Coherency is regarded as a viability constraint for the differential inclusions describing the evolutionary games. Therefore, we investigate viability properties of the constraints and characterize velocities of pairs belief/strategies which guarantee that coherency of beliefs is always satisfied. Finally, among many coherent belief revisions (evolutions), we investigate those characterized by minimal change and provide existence results.

2. The stage game

Let $I = \{1,\dots,n\}$ be the set of players. Then the *coalition structures set* (*CS set*, for short) is the set $\Im$ of all partitions of *I*. In Hart and Kurz (1983), the strategy set of each agent *i* is $\Sigma_i = \{S \subseteq I \mid i \in S\}$, so that a strategy profile is the *n*-tuple $(S_1,\dots,S_n) \in \prod_{i\in I} \Sigma_i$ and the strategy S_i is the set of players that player *i* wishes to join. The γ and the δ rules of coalition formation specify which coalition structures form for every strategy profile. Such rules are given by the functions $h^{\gamma}:\prod_{i\in I}\Sigma_i \to \Im$; $h^{\delta}:\prod_{i\in I}\Sigma_i \to \Im$ defined respectively by

1) $T\in h^{\gamma}(S_1,\ldots,S_n) \Leftrightarrow \quad T=S_j$ for all $j\in T$ or $T=\{l\}$ for some $l\in I$
2) $T\in h^{\delta}(S_1,\ldots,S_n) \Leftrightarrow \quad S_i=S_j$ for all $i,j\in T$ or $T=\{l\}$ for some $l\in I$

2.1 Mixed Strategies

Differently from Hart and Kurz (1983), we assume that every player is allowed to choose a mixed strategy, called mixed coalition, i.e. a vector of probabilities $m_i=(m_{i,S})_{S\ni i}$ such that $m_{i,S}\geq 0$ for all $S\ni i$ and $\sum_{S\ni i} m_{i,S}=1$.

We notice that, given a mixed strategy profile, the probability of each coalition is not unequivocally determined so we propose two rules of mixed coalition formation which provide a probability assignment on coalitions $(\mu_S)_{S\subseteq I}$ and generalize the γ and the δ rules:

$$\gamma:=\begin{cases}\mu_S=\prod_{i\in S} m_{i,S} & \text{for } |S|\geq 2\\[2ex] \mu_{\{i\}}=1-\sum_{S\ni i,|S|\geq 2}\mu_S & \text{for every } i\in I\end{cases}\qquad;\qquad \delta:=\begin{cases}\mu_S=\max_{T\supseteq S}\left[\prod_{i\in S} m_{i,T}\right] & \text{for } |S|\geq 2\\[2ex] \mu_{\{i\}}=1-\sum_{S\ni i,|S|\geq 2}\mu_S & \text{for every } i\in I\end{cases}$$

We refer to the previous rules as the mixed γ and the mixed δ rules. We notice that these rules satisfy: $\sum_{S\ni i}\mu_S=1$ for all $i\in I$. This means that the restriction of the probability assignment to the strategy set of each player is a probability distribution which, however, might be different from the his mixed strategy.

2.1 Coherent Coalition Structures Beliefs

Since players give probabilities to the coalitions they wish to join, then the outcome of their play is uncertain. To describe this uncertainty, we call *coalition structure belief* (*CS belief* for short) a probability distribution $\varrho=(\varrho_{\mathcal{B}})_{\mathcal{B}\in\Im}$ on $\Im$. A coalition S can be interpreted as an event in the set of all coalition structures $\Im$, more precisely $S=\{\mathcal{B}\in\Im \mid S\in\mathcal{B}\}$ for every $S\subseteq I$, so that the probability μ_S can be regarded as $\mu_S = prob\{\mathcal{B}\in\Im \mid S\in\mathcal{B}\}$ for every $S\subseteq I$.

We require that a CS belief should be coherent in the sense of de Finetti (1931) with the probability assignments on the event/coalition S for every $S\subseteq I$. This is equivalent to say that, for every event/coalition S, the total probability theorem should be satisfied, that is, a CS belief must satisfy the following *coherency constraints:*

(CC) := $\sum_{\mathcal{B}\ni S}\varrho_{\mathcal{B}}=\mu_S \quad \forall S\subseteq I$.

We recall that the total number of partitions of a set with n elements is given by the *Bell number*, denoted with $B(n)$, and defined by the following recursive equation $B(n)=\sum_{k=0}^{n-1}\binom{n-1}{k}B(k)$.

Moreover, the number of non empty coalitions of a set with n elements is $C(n)=\sum_{k=1}^{n}\binom{n}{k}=2^n-1$.

Therefore, coherency constraints (CC) define a linear system with $C(n)$ equations in the $B(n)$ unknowns $\varrho = (\varrho_{\mathcal{B}})_{\mathcal{B}\in\Im}$. We prove that:

i. Whenever there are more than 5 players $B(n)$ is larger than $C(n)$.
ii. Whenever there are more than 5 players, coherency constraints (CC) might have more than one solution.
iii. Whenever the probability assignment $(\mu_S)_{S\subseteq I}$ is given by the the mixed γ or the mixed δ rules, then the system (CC) admits a least a solution.

3. Dynamics

We analyze the evolutionary games arising from the mixed γ and the mixed δ models. We consider the situation in which players determine at every instant the set of players they wish to join, more precisely players act on the velocities of the strategies which are regarded as controls, and then, we study how the coalition structure belief might evolve, governed by Nature, according to these strategic choices, that is, coherently with the probability assignments on coalitions determined by the strategies. So we look at the solutions of the following control system subject to state constraints (K):

$$\begin{cases} \varrho_{\mathcal{B}}'(t) = h_{\mathcal{B}}(t) \quad \forall \mathcal{B}\in\Im \\ m_i'(t) = u_i(t), \quad \forall i\in I \\ u_i(t)\in U_i(m(t),\varrho(t)) \quad \forall i\in I \\ (h_{\mathcal{B}})_{\mathcal{B}\in\Im}\in H(m(t)),\varrho(t)) \end{cases} ; \qquad (K):= \begin{cases} i)\varrho_{\mathcal{B}}\geq 0\,\forall\mathcal{B}\in\Im \\ ii)\sum_{\mathcal{B}\in\Im}\varrho_{\mathcal{B}} - 1 = 0 \\ iii)m_{i,S}\geq 0\,\forall S\subseteq I \text{ with } S\ni i, \forall i\in I \\ iv)\sum_{S\ni i} m_{i,S} - 1 = 0 \quad \forall i\in I \\ v)\chi_S(\varrho,m) = \sum_{\mathcal{B}\ni S}\varrho_{\mathcal{B}} - \mu_S = 0 \quad \forall S\subseteq I \end{cases}$$

Of course, there is no a-priori reason why a solution of the system should be *viable* in the constraints K for every $t\in[0,+\infty[$, that is, satisfy the constraints for every $t\in[0,+\infty[$. Moreover, the constraints set K is said to be *viable* under the control system if for every point in K there exists at least one solution starting from this point which remains in K for every $t\in[0,+\infty[$. So we are interested to characterize velocities such that the corresponding solutions and the set K are viable. To this purpose we apply the main viability theorems for control systems as stated in Aubin (1991, 1997).

Main result

Recall that a direction v belongs to the *contingent cone* $T_K(y)$ of the set K in a point $y\in K$ if there exist sequences $\varepsilon_n > 0$ converging to 0 and v_n converging to v such that $y+\varepsilon_n v_n \in K \quad \forall n\in\mathbb{N}$.

Denote $\mathcal{M}(\varrho,m) = H(\varrho,m)\times\left[\prod_{i=1}^{n} U_i(\varrho,m)\right]$ then we define the *regulation map* as

$R_K(\varrho,m) = \{(h,u)\in\mathcal{M}(\varrho,m) \,|\, (h,u)\in T_K(\varrho,m)\}$.

So we can prove that under suitable assumption, the set K is viable if and only if the images of the regulation map are not empty. Moreover every viable evolution of pairs CS belief/mixed strategies is governed by the following system: $\begin{cases} \varrho_{\mathcal{B}}'(t) = h_{\mathcal{B}}(t) \quad \forall\mathcal{B}\in\Im \\ m_i'(t) = u_i(t), \quad \forall i\in I \\ (u(t),h(t))\in R_K(\varrho(t),m(t)). \end{cases}$

where the regulation map, in the the mixed γ model, has to satisfy the following

$$\begin{cases} i) \quad h_{\mathcal{B}} \geq 0 \quad \text{whenever } \varrho_{\mathcal{B}} = 0 \\ ii) \quad \sum_{\mathcal{B} \in \mathfrak{I}} h_{\mathcal{B}} = 0 \\ iii) \quad u_{i,S} \geq 0 \quad \text{whenever } m_{i,S} = 0 \\ iv) \quad \sum_{S \ni i} u_{i,S} = 0, \quad \forall i \in I \\ v) \quad \sum_{\mathcal{B} \ni S} h_{\mathcal{B}} - \sum_{i \in S} u_{i,S} \left(\prod_{j \in S \setminus \{i\}} m_{j,S} \right) = 0, \quad \forall S \subseteq I, |S| \geq 2 \\ vi) \quad \sum_{S \not\supseteq \{k\}} \left[\sum_{i \in S} u_{i,S} \left(\prod_{j \in S \setminus \{i\}} m_{j,S} \right) \right] + \sum_{\mathcal{B} \ni \{k\}} h_{\mathcal{B}} = 0, \quad \forall k \in I. \end{cases}$$

while, in the mixed δ model, has to satisfy:

$$\begin{cases} i) \quad h_{\mathcal{B}} \geq 0 \quad \text{whenever } \varrho_{\mathcal{B}} = 0 \\ ii) \quad \sum_{\mathcal{B} \in \mathfrak{I}} h_{\mathcal{B}} = 0 \\ iii) \quad u_{i,S} \geq 0 \quad \text{whenever } m_{i,S} = 0 \\ iv) \quad \sum_{S \ni i} u_{i,S} = 0, \quad \forall i \in I \\ v) \quad \sum_{\mathcal{B} \ni S} h_{\mathcal{B}} - \sum_{i \in S} u_{i,S} \left(\prod_{j \in S \setminus \{i\}} m_{j,S} \right) = 0, \quad \forall S \subseteq I, |S| \geq 2 \\ vi) \quad \sum_{S \not\supseteq \{k\}} \left[\sum_{i \in S} u_{i,S} \left(\prod_{j \in S \setminus \{i\}} m_{j,S} \right) \right] + \sum_{\mathcal{B} \ni \{k\}} h_{\mathcal{B}} = 0, \quad \forall k \in I. \end{cases}$$

Belief Revision

Finally, given feedback (state dependent) controls of the players, the evolution of the beliefs can be regarded as a problem of *probabilistic belief revision*. Namely, the classical question in belief revision theory is the following: Suppose one holds a certain belief about the states of the world and at a given moment something that contradicts these belief is observed. How should the belief be revised? We focus on the idea of minimal change revision (see Perea (2007)). if one observes an event that contradicts the previous belief, then the new belief about the world should explain the event just observed, and should be ``as close as possible'' to the previous ones, i.e. previous belief should change, but not more than necessary. In our case, belief revision works in continuous time and revised belief explain observations at every instant through the coherency conditions, since observations are in terms of probability assignments on coalitions rather than events; moreover, the idea of minimal change is translated in terms of revision with minimal velocity and we provide existence theorems for evolutions of CS belief of minimal velocity.

Bibliography

Aubin J.-P. (1991), Viability Theory, Birkhauser, Boston.
Hart S. and Kurz M. (1983), Endogenous Coalition Formation, Econometrica, 54: 1047-1064.
Perea A. (2007), A Model of Minimal Probabilistic Belief Revision, Theory and Decision, on-line first.