Flow dynamics in financial networks

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Abstract

In this paper we analyse the dynamics of flows of losses and of flows of liquidity in financial networks, using an approach based on the theory of flow networks. To this end, we represent financial networks as flow networks, i.e., directed and weighted graphs endowed with source nodes and sink nodes. We then model the 'domino effect', i.e., the diffusion of losses and defaults along the links of the network, and the transfer of liquidity across interbank networks.

JEL classification: C63, G10, G33.

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1 Financial flow networks

In this paper we put forward a novel approach, based on the theory of *flow networks*,[1] for the analysis of contagion in networks of agents connected among themselves by financial obligations. Financial contagion is broadly defined as the transmission of financial distress across agents, sectors or regions of the economy.[2] To analyse the mechanics of such a type of contagion, we represent a financial network as a flow network, i.e., a directed and weighted graph endowed with source nodes and a sink node. Let $\Omega = \{\omega_i\}, i = 1...n$, be the a set of financial operators. Let $c_{ij} \in \mathcal{R}^+$, be the amount of debt, if any, that agent i owes agent j, and let $C = \{c_{ij}\}$, for $i, j = 1...n$ and $i \neq j$. Each agent in Ω is characterized by its own balance sheet. Let $a_i \in \mathcal{R}^+$ be the value of the *external assets* owned by ω_i–i.e., assets issued by agents that do not belong to Ω–and let $r_i \in \mathcal{R}^+$ be the sum of the loans granted by ω_i to other agents

[1] See Ahuja, K. R., Magnanti, T. L. and Orlin, J. B. (1993) *Network Flows: theory, algorithms and applications*, Prentice Hall, New Jersey, for a reference book on the theory of flows and flow network.

[2] The main theoretical contributions to the analysis of finacial contagion are Allen, F. and Gale, D. (1998) "Optimal Financial Crises", *Journal of Finance*, 53, 1245-48. – (2000) "Financial Contagion", *Journal of Political Economy*, 108(1): 1-34, Freixas, X., Parigi, B. and Rochet, J. C. (1999) "Systemic risk, interbank relations and liquidity provision by the central bank"

in Ω, i.e., $r_i = \sum_j c_{ji}$. On the liability side of the balance sheet, let $d_i \in \mathcal{R}^+$ be the sum of the loans granted to ω_i by other agents in Ω, i.e., $d_i = \sum_j c_{ij}$; and let $a_i + r_i - d_i \equiv v_i \in \mathcal{R}$ be the net worth of the i-th agent. Finally, let $A = \{a^k\}, k = 1...m$, be a set of external assets such that each a^k in A appears in the balance sheet of at least one operator in Ω, and let $a_i^k \in \mathcal{R}^+$ be the amount of asset k held by agent i, if any.

We represent this financial system as a *multisource network*, i.e., a directed and connected graph, with some sources and a sink, whose links are endowed with non-negative capacities. More precisely, let $N = \{\Omega, A, t, \Lambda, S, L, \Gamma\}$ be a multisource network where:

1. $\Omega = \{\omega_i\}$ is the set of n nodes that represent the above defined agents;
2. $A = \{a^k\}$ is the set of m *source nodes*, i.e., nodes with no incoming links, that represent the external assets;
3. t is the *sink*, i.e., a terminal node with no outgoing links;
4. $\Lambda \subseteq \Omega^2$ is a set of ordered pairs of nodes in Ω, i.e., a set of directed links $\{l_{ij}\}$ representing the liabilities in C, where l_{ij} starts from node ω_i and ends in node ω_j, and $l_{ij} \in \Lambda$ only if $c_{ij} > 0$.
5. $S = \left\{s_i^k\right\}$ is a set of directed links, with start nodes in A and end nodes in Ω, that connect the external assets to their owners, where $s_i^k \in S$ only if $a_i^k > 0$.
6. $L = \left\{l_t^i\right\}$ is a set of directed links, with start nodes in Ω and end node t, that connect each node in Ω to the sink.
7. $\Gamma : \Lambda \to \mathcal{R}^+, S \to \mathcal{R}^+, L \to \mathcal{R}^+$ is a map, called *capacity function*, that associates i) to each l_{ij} the value of the corresponding liability c_{ij}, ii) to each s_i^k the value of the corresponding asset a_i^k, and iii) to each l_t^i the net worth, v_i, of its start node ω_i.

We use the above defined network N to model the propagation of financial losses among the agents in Ω,originated by the default of one or more agents, as a flow across N.

The propagation of these losses through the network is governed by the rules of limited liability, debt priority and pro-rata reimbursement of creditors. When a node suffers a loss, this loss is first absorbed by the net worth of the node. Only the residual loss, if any, is passed over to other nodes in Ω. The losses that are offset by the equity of the agents in Ω exit the flow of losses that circulate across the network N. The sink node is the virtual bucket where, for modelling convenience, we direct such losses. For each node ω_i in Ω, let

$$\beta_i(\lambda_i) = \min \left(\frac{\lambda_i}{v_i}, 1\right) \tag{1}$$

be an *activation function*, where λ_i is the total loss born by the i-th node– received from source nodes and/or from other nodes in Ω. If a node ω_i receives a positive flow of losses, it is activated and sends to the sink an amount of its own net worth equal to $\beta_i v_i$. If the losses suffered by ω_i are larger than its net worth, then this node is insolvent and sends the residual loss, $\lambda_i - v_i$, to its creditors, i.e., to its direct descendants in Ω. For each node ω_i in Ω, let

$$b_i(\lambda_i) = \max \left(0, \frac{\lambda_i - v_i}{d_i}\right) \tag{2}$$

be an *insolvency function*. If the operator defaults, i.e. $b_i > 0$, its assets are liquidated and its creditors get a *pro rata* refund. We assume that this is done without delays and without incurring bankruptcy costs.[3] Thus the loss born by an agent is

$$\lambda_i = \sum_k b_k a_i^k + \sum_j b_j c_{ji}.$$

We show that the iterated application of (1) and (2), following a shock, yields a flow (of losses) defined in a flow network N. We then use the properties of network flows to investigate the relation between some characteristics of a financial networks and its resiliency towards default contagion. The results we obtained show the existence of a relation between the threshold of default contagion, i.e., the magnitude of the smallest shock capable of causing further defaults, and the scope of contagion, i.e., the number of defaulting agents. The network structure that ensures the most uniform distribution of losses among the agents is the complete network, where everybody lends to everybody else. This structure has the highest contagion threshold and scope, in the sense that it is resilient to default contagion for small-medium size shocks while, in case of a large enough shock, the contagion is maximal: the entire network defaults. The star-shaped money centre structure displays similar features. Conversely, an incomplete network structure, where each agent is connected to a limited number of other agents, has contagion threshold and scope lower than the ones of a complete structure: small shocks can cause default contagion involving a limited number of agents. Thus regulators face a trade-off in designing optimal financial structures: the choice between structures exposed to low probability-high impact events, such as the complete and star-shaped networks, and structures exposed to high probability-low impact events, such as the incomplete networks.

1.1 Interbank liquidity networks

We apply the present flow network approach also to the analysis of liquidity transfers in interbank networks. We suppose that the network is in a balanced initial state, where no bank experiences a liquidity shortage or surplus. We then perturb the system by a liquidity shock that consists of a reallocation of customer deposits across banks, with no aggregate liquidity shortage (for the time being). Formally, a *liquidity shock* is an ordered vector of scalars $\boldsymbol{\delta} = [\delta_1, \delta_2, ..., \delta_n]$, where $\sum_{\Omega_b} \delta_i = 0$.

We model the transfer of liquidity from surplus nodes to deficit nodes as a flow across N_b, a flow driven by deposit withdrawals. The liquidity need

[3]Bankruptcy costs can be introduced in the model by adding extra sources of losses that get activated in case of defaults. These extra losses would make the system more prone to widespread crisis without substantially altering the results presented below.

of a bank ω_i is the sum of its customers' withdrawals, δ_i, and of the withdrawals of interbank deposits undertaken by its direct descendants in N_b. Let $G^1(\omega_i) = \{\omega_j | l_{ij} \in \Lambda_b\}$ be the set of children nodes of ω_i, i.e., the set of its direct descendants. Let $\rho_{ij} \in (0, c_{ij})$ measure the interbank deposits withdrawn by node $\omega_j$from its parent node ω_i . The *liquidity shortage* faced by a bank is

$$\sigma_i = \delta_i + \sum\nolimits_{G^1(\omega_i)} \rho_{ij}.$$

When σ_i takes on a positive value, node ω_i faces a liquidity shortage and it first withdraws part or all of its deposits from its parent nodes. Then, if $\sigma_i > r_i$, ω_i resorts to the liquidation of external (possibly illiquid) assets a_i to cover the residual liquidity need. We assume that banks in liquidity deficit withdraw interbank deposits in a *pro rata* fashion. Formally, we assume: $\rho_{ji} = \eta_i c_{ji}$, where ρ_{ji} is the amount of money withdrawn by ω_i from ω_j and

$$\eta_i(\sigma_i) = \min\left[\max\left(\frac{\sigma_i}{r_i}, 0\right), 1\right] \tag{3}$$

is the *withdrawal function* that associates the quota $\eta_i \in (0, 1)$ of the proportional deposit withdrawal undertaken by node ω_i, to its own liquidity shortage, σ_i. If $\sigma_i > r_i$, the short-term interbank exposures of node ω_i are insufficient to satisfy its deficit and it must liquidate part or all of its external assets. Let

$$\theta_i(\sigma_i) = \max\left[0, \frac{\sigma_i - r_i}{a_i}\right] \tag{4}$$

be the *liquidation function* that yields the portion θ_i of external assets a_i to be liquidated.

When the initial liquidity shock occurs, the nodes in deficit withdraw their interbank deposits according to (3) and, if necessary, liquidate external assets according to (4). By withdrawing their deposits, the banks in deficit transfer their liquidity shortage (or part of it, if asset liquidation is needed) to their parent nodes which, in turn, do the same to their own parent nodes, and so forth until all nodes in the network have achieved a balanced liquid position. We show that the iterated application of (3) and (4), following a liquidity shock, generates a flow (of liquidity) defined in a flow network N_l. We then compare the transfer capacity of complete, incomplete and star-shaped interbank networks. We show that a) cycle flows in interbank transfers cause a waste of the transfer capacity of a network, therefore they reduce the resiliency of a network towards liquidity shocks; b) larger interbank exposures, on one hand, render the network more exposed to default contagion and, on the other hand, render the banking system more resilient to liquidity shocks. Finally, we compare the transfer capacity of complete, incomplete and star-shaped interbank networks, we show that star-shaped networks are the most effective in reallocating liquidity across banks.

