An unified approach to the pairwise comparison matrices

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Abstract

We present a general approach to pairwise comparison matrices and introduce a consistency index that is easy to compute in the additive and multiplicative case; in the other cases it can be computed easily starting from a suitable additive or multiplicative matrix.

1 Introduction

Let $X = \{x_1, x_2, ..., x_n\}$ be a set of alternatives or criteria. An useful tool to determine a weighted ranking on X is a *pairwise comparison matrix* (PCM for short)

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{1.1}$$

which entry a_{ij} expresses how much the alternative x_i is preferred to alternative x_j. A condition of *reciprocity* is assumed for the matrix $A = (a_{ij})$ in such way that the preference of x_i over x_j expressed by a_{ij} can be exactly read by means of the element a_{ji}. Under a suitable condition of *consistency* for $A = (a_{ij})$, X is totally ordered and there exists a *consistent vector* $\underline{w}$, that perfectly represents the preferences over X; then $\underline{w}$ provides the proper weights for the the elements of X.

The shape of the reciprocity and consistency conditions depend on the different meaning given to the number a_{ij}, as the following well known cases show.

Multiplicative case: $a_{ij} \in]0, +\infty[$ is a preference ratio and the conditions of reciprocity and consistency are given respectively by

$$\textbf{mr)} \quad a_{ji} = \frac{1}{a_{ij}} \quad \forall\, i, j = 1, \dots, n \qquad \textit{(multiplicative reciprocity)},$$

$$\textbf{mc)} \quad a_{ik} = a_{ij} a_{jk} \quad \forall\, i, j, k = 1, \dots, n \qquad \textit{(multiplicative consistency)}.$$

A consistent vector is a positive vector $\underline{w} = (w_1, w_2, ..., w_n)$ verifying the condition $\frac{w_i}{w_j} = a_{ij}$.

Additive case: $a_{ij} \in] - \infty, +\infty[$ is a preference difference and reciprocity and consistency are expressed as follows

$$\textbf{ar)} \quad a_{ji} = -a_{ij} \quad \forall\, i, j = 1, \dots, n \qquad \textit{(additive reciprocity)},$$

$$\textbf{ac)} \quad a_{ik} = a_{ij} + a_{jk} \quad \forall\, i, j, k = 1, \dots, n \qquad \textit{(additive consistency)}.$$

A consistent vector is a vector $\underline{w} = (w_1, w_2, ..., w_n)$ verifying the condition $w_i - w_j = a_{ij}$.

Fuzzy case: $a_{ij} \in [0, 1]$ measures the distance from the indifference that is expressed by 0.5; the conditions of reciprocity and consistency are the following

$$\textbf{fr)} \quad a_{ji} = 1 - a_{ij} \quad \forall\, i, j = 1, \dots, n \qquad \textit{(fuzzy reciprocity)},$$

$$\textbf{fc)} \quad a_{ik} = a_{ij} + a_{jk} - 0.5 \quad \forall\, i, j, k = 1, \dots, n \qquad \textit{(fuzzy consistency)}.$$

A consistent vector is a vector $\underline{w} = (w_1, w_2, ..., w_n)$ verifying the condition $w_i - w_j = a_{ij} - 0.5$.

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The multiplicative *PCMs* play a basic role in the Analytic Hierarchy Process, a procedure developed by T.L. Saaty at the end of the 70s ([8], [9]), and widely used by governments and companies ([9], [11], [6]) in fixing their strategies. Saaty indicates a scale translating the comparisons expressed in verbal terms into the preference ratios a_{ij}. By applying this scale, a_{ij} may only take value in $S^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$. Actually the Saaty scale restricts the decision maker's possibility to be consistent: indeed if he expresses the preference ratios $a_{ij} = 5$ and $a_{jk} = 3$ then he will not be consistent because $a_{ij}a_{jk} = 15 > 9$. Analougsly, under the assumption that $a_{ij} \in [0, 1]$, the fuzzy consistency property **fc** cannot be respected by a decision maker who claims $a_{ij} = 0.9$ and $a_{jk} = 0.8$, because $a_{ij} + a_{jk} - 0.5 = 1.7 - 0.5 > 1$. A measure of closeness to the consistency for a multiplicative *PC* matrix has been provided by Saaty in terms of the principal eigenvalue λ_{max} [9], [10]. This measure has been questioned because it is not easy to compute, has not a simple and geometric meaning [7], [3] and, in some cases, seems to be unfair [4]. Also the methods used to provide a weighted ranking in case of inconsistency have been questioned: indeed they may indicate rankings that do not agree with the expressed preference ratios a_{ij} [1], [2].

We present a general framework for *PCMs*, in which the entry a_{ij} of the matrix belongs to a set G structured as abelian linearly ordered group in such way that the consistency drawback is removed. We provide also a consistency index that is naturally grounded on a notion of distance and is easy to compute in the case of multiplicative or additive matrix.

2 Alo-groups

Let G be a non empty set provided with a total weak order $\leq$ and a binary operation $\odot : G \times G \to G$. $\mathcal{G} = (G, \odot, \leq)$ is called *abelian linearly ordered* group *(alo-group* for short), if and only if $(G, \odot)$ is an abelian group and the the following implication holds:

$$a < b \Rightarrow a \odot c < b \odot c,$$

where $<$ is the strict simple order associated to $\leq$.
If $\mathcal{G} = (G, \odot, \leq)$ is an alo-group, then we will assume that: e denotes the *identity* of $\mathcal{G}$, $x^{(-1)}$ the *symmetric* of $x \in G$ with respect to $\odot$, $\div$ the *inverse operation* of $\odot$ defined by "$a \div b = a \odot b^{(-1)}$".
For a positive integer n, the *(n)-power* $x^{(n)}$ of $x \in G$ is defined as follows

$$x^{(1)} = x \qquad \textit{and} \qquad x^{(n)} = \bigodot_{i=1}^{n} x_i, \; x_i = x \;\; \forall i = 1, ..., n, \text{for } n \geq 2.$$

If $b^{(n)} = a$, then we say that b is the *(n)-root* of a and write $b = a^{(1/n)}$. $\mathcal{G}$ is *divisible* if and only if for each positive integer n and each $a \in G$ there exists the (n)-root of a.

Proposition 2.1. *A non trivial alo-group $\mathcal{G} = (G, \odot, \leq)$ has neither the greatest element nor the least element.*

So, by Proposition 2.1, neither the interval [0, 1] nor the Saaty set $S^* = \{1, \ldots, 9, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}\}$, embodied with the usual order $\leq$ on R, can be structured as alo-group.

Proposition 2.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the operation*

$$d_{\mathcal{G}} : (a, b) \in G^2 \to d_{\mathcal{G}}(a, b) = ||a \div b|| = (a \div b) \vee (b \div a) \in G \tag{2.1}$$

verifies the conditions:

1. $d_{\mathcal{G}}(a, b) \geq e$ *and* $d_{\mathcal{G}}(a, b) = e \Leftrightarrow a = b$;
2. $d_{\mathcal{G}}(a, b) = d_{\mathcal{G}}(b, a)$;
3. $d_{\mathcal{G}}(a, b) \leq d_{\mathcal{G}}(a, c) \odot d_{\mathcal{G}}(b, c)$.

Definition 2.1. *The operation $d_{\mathcal{G}}$ in* (2.1) *is a $\mathcal{G}$*-metric *or $\mathcal{G}$*-distance.

Definition 2.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be a divisible alo-group. Then, the $\odot$*- mean $m_{\odot}(a_1, a_2, ..., a_n)$ *of the elements $a_1, a_2, ..., a_n$ of G is defined by*

$$m_{\odot}(a_1, a_2, ..., a_n) = \begin{cases} a_1 & \textit{for n=1 ,} \\ (\bigodot_{i=1}^{n} a_i)^{(1/n)} & \textit{for } n \geq 2. \end{cases}$$

Isomorphisms between alo-groups An *isomorphism* between two alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ is a bijection $h : G \to G'$ that is both a lattice isomorphism and a group isomorphism, that is:

$$x < y \Leftrightarrow h(x) < h(y) \quad and \quad h(x \odot y) = h(x) \circ h(y).$$

Proposition 2.3. *Let $h : G \to G'$ be an isomorphism between the alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$. Then,*

$$d_{\mathcal{G}'}(a', b') = h(d_{\mathcal{G}}(h^{-1}(a'), h^{-1}(b'))), \quad d_{\mathcal{G}}(a, b) = h^{-1}(d_{\mathcal{G}'}(h(a), h(b))).$$

Moreover, $\mathcal{G}$ is divisible if and only if $\mathcal{G}'$ is divisible and, under the assumption of divisibility:

$$m_\circ(y_1, y_2, ..., y_n) = h\big(m_\odot(h^{-1}(y_1), h^{-1}(y_2), ..., h^{-1}(y_n))\big).$$

Real alo-groups An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R. Let $+$ and $\cdot$ be the usual addition and multiplication on R and $\otimes :]0,1[^2 \to]0,1[$ the operation defined by $x \otimes y = \frac{xy}{xy+(1-x)(1-y)}$. Then examples of real divisible alo-groups are the following:

Multiplicative alo-group: $]\mathbf{0,+\infty}[= (]0,+\infty[, \cdot, \leq)$; then $e = 1$, $x^{(-1)} = x^{-1} = 1/x$, $x^{(n)} = x^n$ and $x \div y = \frac{x}{y}$. So $d_{]\mathbf{0,+\infty}[}(a, b) = \frac{a}{b} \vee \frac{b}{a}$ and $m_\cdot(a_1, ..., a_n)$ is the geometric mean: $\big(\prod_{i=1}^n a_i\big)^{\frac{1}{n}}$.

Additive alo-group: $\mathcal{R} = (R, +, \leq)$; then $e = 0$, $x^{(-1)} = -x$, $x^{(n)} = nx$, $x \div y = x - y$. So $d_{\mathcal{R}}(a, b) = |a - b| = (a - b) \vee (b - a)$ and $m_+(a_1, ..., a_n)$ is the arithmetic mean: $\frac{\sum_i a_i}{n}$.

Fuzzy alo-group: $]\mathbf{0,1}[= (]0,1[, \otimes, \leq)$; then $e = 0.5$, $x^{(-1)} = 1 - x$, $x \div y = \frac{x(1-y)}{x(1-y)+(1-x)y}$ and $d_{]\mathbf{0,1}[}(a, b) = \frac{a(1-b)}{a(1-b)+(1-a)b} \vee \frac{b(1-a)}{b(1-a)+(1-b)a}$.

The above alo-groups are isomorphic: $h : x \in]0,+\infty[\to \log x \in R$ is an isomorphism between $]\mathbf{0,+\infty}[$ and $\mathcal{R}$ and $v : t \in]0,+\infty[\to \frac{t}{t+1} \in]0,1[$ is an isomorphism between $]\mathbf{0,+\infty}[$ and $]\mathbf{0,1}[$. So, by Proposition 2.3, the mean $m_\otimes(a_1, ..., a_n)$ related to the fuzzy alo-group can be computed as follows: $m_\otimes(a_1, ..., a_n) = v\big(\big(\prod_{i=1}^n v^{-1}(a_i)\big)^{\frac{1}{n}}\big)$.

3 Pairwise comparison matrices over a divisible alo-group

In this section we assume that $\mathcal{G} = (G, \odot, \leq)$ is divisible alo-group. A pairwise comparison system over $\mathcal{G} = (G, \odot, \leq)$ is a pair $(X, \mathcal{A})$ constituted by a set $X = \{x_1, ..., x_n\}$ and a relation $\mathcal{A} : (x_i, x_j) \in X^2 \to a_{ij} = \mathcal{A}(x_i, x_j) \in G$. The relation $\mathcal{A}$ is represented by means of the matrix in (1.1) with entries a_{ij} belonging to G. We say that $A = (a_{ij})$ is a *PCM* over $\mathcal{G}$ and assume that A that is *reciprocal* with respect to $\odot$, that is :

$$\mathbf{r}_\odot) \quad a_{ji} = a_{ij}^{(-1)} \quad \forall\, i, j = 1, \ldots, n \qquad (\odot\text{-}reciprocity)$$

so $a_{ii} = e$ for each $i = 1, 2, ..., n$ and $a_{ij} \odot a_{ji} = e$ for $i, j \in \{1, 2, ..., n\}$.

Let $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ be the rows of $A = (a_{ij})$; then the *mean vector* associated to A is the vector

$$\underline{w}_{m_\odot}(A) = (m_\odot(\underline{a}_1), m_\odot(\underline{a}_1), \cdots, m_\odot(\underline{a}_n)). \tag{3.1}$$

Definition 3.1. *$A = (a_{ij})$ is a* consistent *matrix with respect to $\odot$, if and only if:*

$$\mathbf{c}_\odot) \quad a_{ik} = a_{ij} \odot a_{jk} \quad \forall i, j, k \qquad (\odot\text{-consistency}).$$

$\underline{w} = (w_1, \ldots, w_n)$ is a consistent *vector for $A = (a_{ij})$ if and only if $w_i \div w_j = a_{ij}$ $\forall$ i, j=1,2,...,n.*

Remark 3.1. *As $\odot$ is an group operation, $a_{ij} \odot a_{jk} \in G$ for every choice of a_{ij} and a_{jk} in G. So the decision maker has the possibility to be consistent and do not fall into the consistency drawback discussed in Section 1.*

Proposition 3.1. *$A = (a_{ij})$ is a* consistent *matrix with respect to $\odot$, if and only if:*

$$d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) = e \text{ for each triple } \quad (i, j, k) \text{ with } i < j < k.$$

Proposition 3.2. *Let $A = (a_{ij})$ be consistent. Then each column $\underline{a}^k$ of A and the mean vector $\underline{w}_{m_\odot}$ in (3.1) are consistent vectors.*

Consistency index Let T be the set $= \{(a_{ij}, a_{jk}, a_{ik}), i < j < k\}$ and $n_T = |T|$. By Proposition 3.1 $A = (a_{ij})$ is inconsistent if and only if $d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) > e$ for some triple $(a_{ij}, a_{jk}, a_{ik}) \in T$. So we give the following definition:

Definition 3.2. *The* consistency index *of $A = (a_{ij})$ is given by*

$$I_{\mathcal{G}}(A) = d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) \qquad \text{if } n = 3;$$

$$I_{\mathcal{G}}(A) = \left(\bigodot_{i<j<k} d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) \right)^{(1/n_T)} \qquad \text{if } n > 3.$$

Proposition 3.3. *$I_{\mathcal{G}}(A) \geq e$ and A is consistent if and only if $I_{\mathcal{G}}(A) = e$.*

Finally, by Proposition 2.3 we get the following result.

Proposition 3.4. *Let $\mathcal{G}' = (G', \circ, \leq)$ be a divisible alo-group isomorphic to $\mathcal{G}$ and $A' = (h(a_{ij}))$ the transformed of $A = (a_{ij})$ by means of the isomorphism $h : G \to G'$. Then $I_{\mathcal{G}}(A) = h^{-1}(I_{\mathcal{G}'}(A'))$.*

References

[1] Basile, L. and D'Apuzzo L. (2002). Weak Consistency and Quasi-Linear Means Imply the Actual Ranking. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, World Scientific Publishing, 10, n. 3, 227-239.

[2] Basile, L. and D'Apuzzo L. (2006). Transitive Matrices, Strict Preference and Ordinal evaluation Operators. *Soft Computing - A Fusion of Foundations, Methodologies and Applications*, vol.10, no.10 Springer- Verlag, 933-940

[3] Barzilai J.(1998). Consistency measures for pairwise comparison matrices. *J. MultiCrit. Decis. Anal.* 7, 123-132

[4] Brunelli, M., Fedrizzi M. (2007) Fair consistency evaluation in fuzzy preference relations and in AHP. *Lecture Notes in Computer Science* , Vol. 4693, p. 612-618. , Atti del convegno: "KES 2007" , Vietri sul mare (Salerno), 12th-14th September 2007.

[5] D'Apuzzo L., Marcarelli G., Squillante M. (2007). Generalized Consistency and Intensity Vectors for Comparison Matrices, *International Journal of Intelligent Systems*, vol. 22, n. 12.

[6] Fusco Girard L. and Nijkamp P.(1997). Le valutazioni per lo sviluppo sostenibile della città e del territorio. Franco Angeli, Milano.

[7] Pelàez, J. I., Lamata, M. T.(2003) A new measure of consistency for positive reciprocal matrices. *Computers and Mathematics with Applications*, 46, 18391845

[8] Saaty, T. L. (1977). A scaling method for priorities in hierarchical structures. *J. Math.Psychology,* 15, 234-281

[9] Saaty, T. L. (1980). The Analytic Hierarchy Process. McGraw-Hill. New York.

[10] Saaty, T. L. (1986). Axiomatic Foundation of the Analytic Hierarchy Process. *Management Sciences*, 32, 841-855.

[11] Saaty, T. L. (1988). Decision Making for Leaders. University of Pittsburgh, Pittsburgh.